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Investigating efficiency of Adomian decomposition method in solving van der pol's equation compared to regular perturbation method

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Abstract
Many physical systems are mathematically modeled leading to nonlinear ordinary differential equations or partial differential equations, raising the need for an effective method for analyzing the mathematical models which provide solutions that conform to physical problems. The Adomian Decomposition Method (ADM) has been used to solve a wide range of dynamical systems since its introduction in 1980's. Nonlinear oscillatory differential equations have been used in modeling many dynamical systems and they demonstrate many basic properties of nonlinear systems. These equations have been solved using many approximations methods e.g. Differential transform, perturbation, variation iteration, and Lindstedt methods. This work investigates the efficiency in the application of ADM versus perturbation method in solving one of the nonlinear oscillatory differential equations, the Van Der Pol's equation. For analysis of accuracy, Runge- Kutta fifth order method is used as a comparison criterion and respective error bounds are obtained. These results will enhance confidence in the application of ADM or perturbation method in solving nonlinear oscillatory systems.

Keywords: adomian decomposition, Perturbation method, van der pol's equation

1. Introduction

In 1984 George Adomian developed a new method of solving nonlinear differential equations. The method was named Adomian Decomposition Method (ADM). ADM requires less computational work compared to numerical methods. It can solve nonlinear problems without linearization hence it provides more reliable analytic solution. This method has been applied by many researchers in solving nonlinear equations for example [1, 8, 2, 4]. The ADM is applied by separating the equation under investigation into linear and nonlinear parts. The higher order derivative of the linear part is inverted and the inverse operator is then applied to the equation. Using the initial or boundary conditions the first term of the series is identified and the nonlinear portion is decomposed into a series of polynomials called Adomian polynomials [3]. The successive terms of the series solution are found by recurrent relation using Adomian's polynomials. The convergence of this method has been proved [5]. Perturbation methods allow the simplification of complex mathematical problems [9]. These methods finds approximate solution by taking advantage of the small parameter that appears in the initial value problem. Numerical methods that closely approximate nonlinear problems are relied on by most mathematicians. This is due to advancement in the field of computational mathematics, where computers of today can solve extremely complex mathematical problems. Nonlinear oscillatory systems have been used in many areas of physics and engineering. These systems are important in mechanical and structural dynamics. Many practical engineering components are made of vibrating systems which can be modeled using oscillator systems. Computational mathematics have really advanced, the performance and accuracy of any new method or modification to the existing ones are investigated by applying them to physical models which are well developed. The Van Der Pol's equation which is a nonlinear oscillator is such a model. It is used to test efficiency and reliability of new methods for solving nonlinear differential equations [6]. In this work the Van Der Pol's equation is used.

2. Application of ADM and Perturbation methods to Van Der pol's equation

The equation is solved using the ADM and the solutions are compared to those evaluated through regular perturbation method. Error analysis is done using solution of Runge- Kutta fifth order of the same equation, which is assumed to be very close to exact solution [7]. The results of the comparison show whether ADM is a more powerful method for solving this equation compared to perturbation method. Consider the unforced Van Der Pol's equation

$$\frac{d^2y}{dt^2} + y - \varepsilon(1 - y^2)\frac{dy}{dt} = 0 \quad (1)$$

Where ε is a positive constant which is a control parameter that reflects the degree of nonlinearity of the system. In studying the case $\varepsilon \ll 1$. y describes the current in a triode oscillator with initial conditions.

$$y(0) = 1 \quad \text{And} \quad y'(0) = 0$$

2.1 Solution of Van Der Pol's Equation Using ADM Method

Rearranging equation yields

$$\frac{d^2y}{dt^2} = -y + \varepsilon\frac{dy}{dt} - \varepsilon y^2\frac{dy}{dt} \tag{2}$$

Taking

$$3y^2\frac{dy}{dt} = \frac{d}{dt}(y^3) \tag{3}$$

and substituting (3) into (2) yields

$$\frac{d^2y}{dt^2} = -y + \varepsilon\frac{dy}{dt} - \frac{\varepsilon}{3}\frac{d}{dt}(y^3) \tag{4}$$

Rewriting the above equation in operator form it transforms to

$$Ly = -y + \varepsilon Ry - \frac{\varepsilon}{3}Ny \tag{5}$$

Where L is the second order differential operator, N is the non linear term $\frac{d}{dt}(y^3)$ and R is $\frac{d}{dt}$. Therefore inverse operator L^{-1} is twofold integration with respect to *t* from *0* to *t*. Applying the inverse operator yields,

$$y(t) = ty'(0) + y(0) - L^{-1}y + \varepsilon L^{-1}Ry - \frac{\varepsilon}{3}L^{-1}Ny \tag{6}$$

Applying the initial conditions in (6) yields

$$y(t) = 1 - L^{-1}y + \varepsilon L^{-1}Ry - \frac{\varepsilon}{3}L^{-1}Ny \tag{7}$$

Approximating

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{8}$$

Decomposing

$$Ny = \frac{d}{dt}\sum_{n=o}^{\infty} A_n \tag{9}$$

Equation (7) becomes

$$\sum_{n=0}^{\infty} y_n = 1 - L^{-1}\sum_{n=0}^{\infty} y_n + \varepsilon L^{-1}\frac{d}{dt}\sum_{n=0}^{\infty} y_n - \frac{\varepsilon}{3}L^{-1}\frac{d}{dt}\sum_{n=0}^{\infty} A_n \tag{10}$$

Using (9) A_n are found to be

$$\begin{aligned} A_0 &= y_0^3 \\ A_1 &= 3y_1y_0^2 \\ A_2 &= 3y_2y_0^2 + 3y_0y_1^2 \\ A_3 &= 3y_3y_0^2 + 6y_0y_1y_2 + y_1^3 \quad \vdots \end{aligned} \tag{11}$$

Using (11) in (9) the recursive relationship that is used to generate the solution

$$y_0(t) = 1 \tag{12}$$

$$y_{n+1} = -L^{-1}y_n + \varepsilon L^{-1}\frac{d}{dt}y_n - \frac{\varepsilon}{3}L^{-1}\frac{d}{dt}A_n \tag{13}$$

The series solution generated by the first five terms is

$$\varphi = \sum_{n=0}^{4} y_n = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{6\varepsilon}{5!}t^5 - \frac{1}{6!}t^6 + \frac{66\varepsilon}{7!}t^7 + \frac{1}{8!}t^8 \tag{14}$$

2.2 Solution of Van der Pol's Equation Using Perturbation Method.

Setting $\varepsilon = 0$, equation (1) becomes

$$\frac{d^2y}{dt^2} + y = 0 \tag{15}$$

This gives the general solution of equation (15) is:

$$y = c_1\cos(t) + c_2\sin(t). \tag{16}$$

Using the initial conditions to solve for c_1 and c_2 yields

$$y = \cos(t) \tag{17}$$

1st order solution of (1).Assuming $\delta(t)$ is a correction factor, the second approximation solution will be

$$y = \cos(t) + \delta(t) \tag{18}$$

Using the initial conditions and substituting the derivatives of (18) in (1) yields

$$-\cos(t) + \frac{d^2\delta(t)}{dt^2} + \cos(t) + \delta(t) = \varepsilon\left[\left(1 - \left(\cos(t) + \delta(t)\right)^2\right)\left(-\sin(t) + \frac{d\delta(t)}{dt}\right)\right] \tag{19}$$

Expanding (19), simplifying and neglecting the higher order terms reduces to

$$\frac{d^2\delta(t)}{dt^2} + \delta(t) = -\varepsilon\sin^3(t) \tag{20}$$

Letting

$$\delta(t) = \varepsilon\psi(t) \tag{21}$$

And substituting (21) into (20) and simplifying yields

$$\frac{d^2\psi(t)}{dt^2} + \psi(t) = -\frac{3}{4}\sin(t) + \frac{1}{4}\sin(3t) \tag{22}$$

The solution of differential equation (22) through super positioning is given by

$$\psi = \psi_{homogenous} + \psi_{particular} \tag{23}$$

The general solution takes the form

$$\psi_{homogenous} = c_3\cos(t) + c_4\sin(t) \tag{24}$$

The particular solution is guessed as

$$\psi_{particlar} = At\sin(t) + Bt\cos(t) + \Gamma\sin(3t) \tag{25}$$

Solving for the coefficients in (24) and (25) using (22), (23) becomes

$$\psi(t) = -\frac{9}{32}\sin(t) + \frac{3}{8}t\cos(t) - \frac{1}{32}\sin(3t) \tag{26}$$

Substituting (26) into (21), and (18) gives the 1st order perturbation approximation solution as

$$y = \cos(t) + \varepsilon\left[-\frac{9}{32}\sin(t) + \frac{3}{8}t\cos(t) - \frac{1}{32}\sin(3t)\right] \tag{27}$$

3. Results Discussion and Conclusion

3.1 Results

Table 1: Absolute errors in Perturbation and ADM solution at $\varepsilon = 0.01$

t	Perturbation	ADM	RK5	Absolute error in perturbation	Absolute error in ADM
0	1	1	1	0	0
0.2	0.9800664195	0.9800664195	0.9800667250	3.055E-07	3.055E-07
0.4	0.9210560842	0.9210560886	0.9210658807	9.97965E-06	9.7921E-06
0.6	0.8253002357	0.8253004024	0.8253709519	7.07162E-05	7.05495E-05
0.8	0.6965681832	0.6965703615	0.6968451170	2.769338E-04	2.747555E-04
1	0.5399177024	0.5399335317	0.5406864807	7.687783E-04	7.52949E-04
1.2	0.3615052921	0.3615845083	0.3632088523	1.703560E-03	1.624344E-03
1.4	0.1683602605	0.1686662918	0.1715705069	3.210246E-03	2.904215E-03
1.6	-0.03187471875	-0.03089745595	-0.0265318001	5.342919E-03	4.365656E-03
1.8	-0.2312331664	-0.2285367478	-0.2231844572	8.048709E-03	5.352291E-03
2	-0.4217380195	-0.4151111111	-0.41057765204	1.116037E-02	4.522459E-03

Table 2: Comparison of absolute errors in Perturbation and ADM solutions at $\varepsilon = 0.5$

T	Perturbation	ADM	Numerical	Absolute error perturbation	Absolute error ADM
0	1	1	1	0	0
0.2	0.9800586612	0.9800586617	0.9800744806	1.58198E-05	1.58189E-05
0.4	0.9208155034	0.9208157217	0.9213055313	4.900279E-04	4.898096E-04
0.6	0.8235666545	0.8235749080	0.8270828332	3.5161787E-03	3.5079252E-03
0.8	0.6897804033	0.6898878740	0.7034465866	1.3666183E-02	1.3558713E-02
1	0.5210721309	0.5218501984	0.5586213214	3.7549105E-02	3.6771123E-02
1.2	0.3197346347	0.3236127451	0.4019090975	8.2174463E-02	7.8296352E-02
1.4	0.08962302108	0.10453974436	0.2423896393	1.5276637E-01	1.3784965E-01
1.6	-0.1629593449	-0.1155524915	0.0879152705	2.5087462E-01	2.0346776E-01
1.8	-0.4287556796	-0.2986397206	-0.0553601456	3.7339553E-01	2.4327956E-01
2	-0.6957059837	-0.3777777778	-0.1829758867	5.1273009E-01	1.9480119E-01

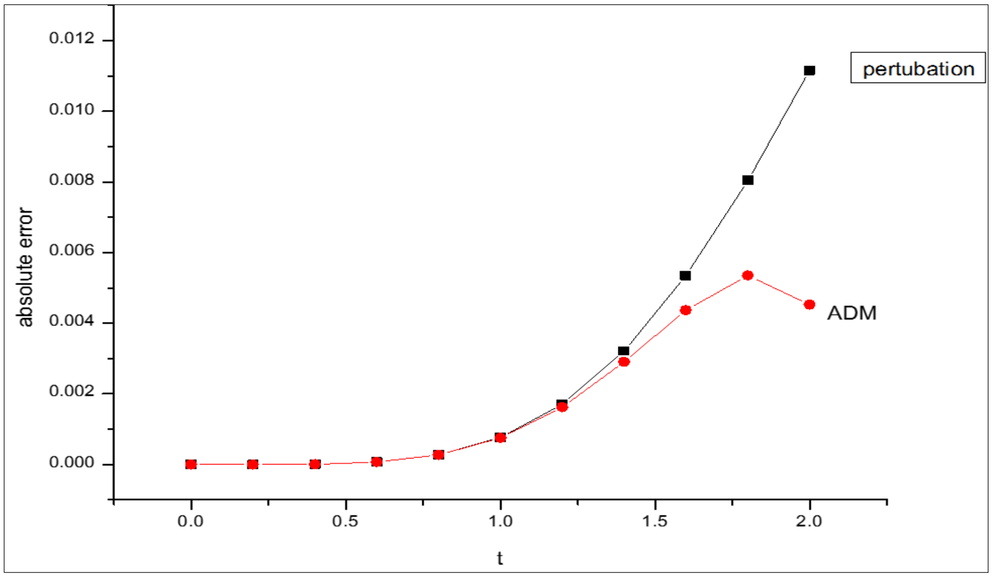

Fig 1: Absolute errors obtained from Perturbation and ADM at $\varepsilon = 0.01$

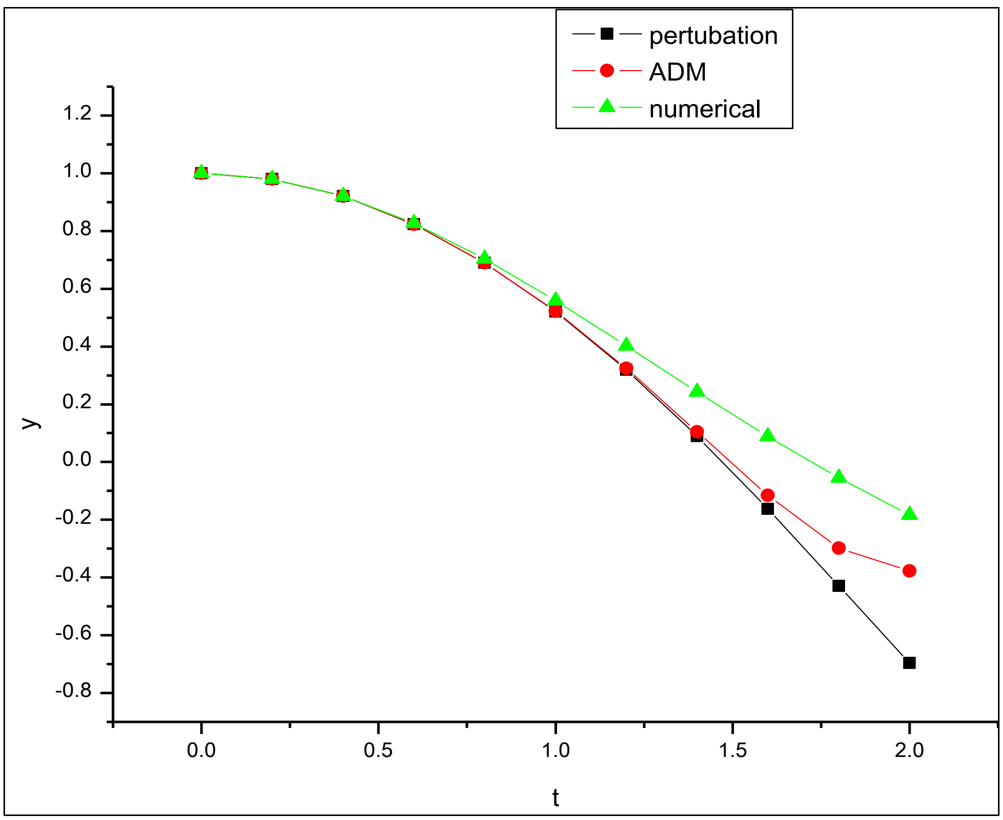

Fig 2: Solution of Van Der Pol's equation using Perturbation, ADM and Numerical method at $\varepsilon = 0.5$

3.2 Discussion and conclusion

The tables and the graphs clearly show that ADM is a very convenient method since it involves direct application, i.e. no linearization or assumptions are made. The computational works are greatly reduced and are easily used in such a way that they can be completed without the assistance of a computer software. The results obtained are quite accurate since they track the numerical solutions with very low errors at low values of *t*. For perturbation method the solution exhibits the periodic behavior but the errors are greater than those of ADM. The computational work is more up to 2nd order, thus 3rd order could be more complicated. Therefore ADM is coming out as more efficient method compared to perturbation method since it produced more accurate solutions within the period $0 \le t \le 2$ but $\varepsilon \ll 1$.

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