International Journal of Advanced Scientific Research
www.allscientificjournal.com
ISSN: 2456-0421
Received: 09-07-2022; Accepted: 26-07-2022; Published: 12-08-2022
Volume 7, Issue 2, 2022, Page No. 20-27

Homomorphisms and automorphisms of product of abelian groups

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Abstract
In this paper, we describe the automorphisms of various classes of finite groups. We introduce the concept of semi direct product on certain groups and use this concept to define the automorphism of certain finite groups. For a semi-direct product of two groups, a short exact sequence splits. A cyclic group of order n is written in the direct product of cyclic $p-$groups. We compute the automorphism groups of some abelian group and also find the number of automorphisms of some abelian groups.

Keywords: homomorphisms, automorphisms

Introduction

The first general structure result for the automorphism group of a finite group follows from a classical result of Gauss in number theory. Let Z_n denotes the additive group of integers $mod\ (n)$. There exist some finite groups that are isomorphic to their own automorphism groups, e.g. D_6. The structure of $Aut(G)$ is often hard to compute. This paper will explore the orders and structures of automorphism groups for direct and semi direct products.

Theorem: Suppose T and S are two subgroups of a group G so that $G = T \rtimes_\tau S,$ where $\tau \in \mathbf{hom}(T, Aut(S))$ $iff\ \exists$ exact short sequence $\{e\} \to S \to G \to T \to \{e\}$ splits.

Proof: Assume $G = T \rtimes_\tau S$, then there is an isomorphism $\lambda : T \rtimes_\tau S \to G.$ A construction for short exact sequence $\{e\} \to T \xrightarrow{\mu} G \xrightarrow{\eta} S \to \{e\}$ and $\xi : T \to G.$ Let construct a map $\mu : T \to G$ and $\xi : S \to G$ by $\mu(t) = \lambda(t, e)$ and $\xi(s) = \lambda(e, s)$ respectively.

Then for $t_1, t_2 \in T$

Let

$$\mu(t_1) = \mu(t_2)$$
$$\Rightarrow \lambda(t_1, e) = \lambda(t_2, e)$$

As λ is 1-1 onto, so

$$t_1 = t_2$$

Hence μ is 1-1.

Also for any $t_1, t_2 \in T$, we get

$$\mu(t_1 t_2) = \lambda(t_1 t_2, e)$$
$$= \lambda(t_1, e)\lambda(t_2, e)$$
$$= \mu(t_1)\mu(t_2).$$

Thus we get μ is 1-1 homomorphism. In the same way, we show that ξ is 1-1 homomorphism.

Each element which belongs to G has formed $\lambda(t, s),$ where $(t, s) \in T \rtimes_\tau S$.

Now, we define $\eta : G \to S$ an onto homomorphism by

$\eta(\lambda(t, s)) = s\ \forall\ \lambda(t, s) \in G.$

Now to prove $im(\mu) = \ker(\eta)$ as well as $(\eta \circ \xi) = id_S.$

Again $\forall\ s \in S,$ we get

$$(\eta \circ \xi)(s) = \eta(\xi(s))$$
$$= \eta(\lambda(t, s))$$
$$= s$$

And

$$
\begin{aligned}
&im(\mu) = \{\lambda(t,e); t \in T\} \\
&= \{\lambda(t,s); s \in S, t \in T \ \&\ \eta(\lambda(s,t)) = e\} \\
&= \ker(\eta).
\end{aligned}
$$

Assuming an exact short sequence $\{e\} \to T \overset{\mu}{\to} G \overset{\eta}{\to} S \to \{e\}$ & a section ξ. Now finding a homomorphism $\tau : S \to Aut(T)$ so that an isomorphism $\Theta : G \to T \rtimes_\tau S$. Suppose an isomorphism $\omega : im(\mu) \to T$ so that $\omega \circ \mu = id_T$ and $\mu \circ \omega = id_{im(\mu)}$.

Now

$$\xi(s)\mu(t)\xi(s^{-1}) \in G \ \forall\ s \in S, t \in T.$$

Hence $\forall\ s \in S,$ an isomorphism $\tau_s : T \to T$ is defined by $\tau_s(t) = \omega(\xi(s)\mu(t)\xi(s^{-1}))$.
Furthermore the mapping $\tau : S \to Aut(T)$ is homomorphism as given by $\tau(s) = \tau_s$ as for any $s_1, s_2 \in S$ and $t \in T$, we get

$$
\begin{aligned}
&(\tau(s_1) \circ \tau(s_2))(t) = (\tau_{s_1} \circ \tau_{s_2})(t) \\
&= (\tau_{s_1}\left(\tau_{s_2}(t)\right) \\
&= \tau_{s_1}\left(\omega(\xi(s_2)\mu(t)\xi(s_2^{-1}))\right) \\
&= \omega\left(\xi(s_1)\mu\left(\omega(\xi(s_2)\mu(t)\xi(s_2^{-1}))\right)\xi(s_1^{-1})\right) \\
&= \omega\left(\xi(s_1)\mu \circ \omega(\xi(s_2)\mu(t)\xi(s_2^{-1}))\xi(s_1^{-1})\right) \\
&= \omega\left(\xi(s_1)(\xi(s_2)\mu(t)\xi(s_2^{-1}))\xi(s_1^{-1})\right) \\
&= \omega(\xi(s_1 s_2)\mu(t)\xi(s_2^{-1}s_1^{-1})) \\
&= \tau_{s_1 s_2}(t) \\
&= \tau(s_1 s_2)(t)
\end{aligned}
$$

Hence for $s_1, s_2 \in S,$ we get $\tau(s_1 s_2) = \tau(s_1)\tau(s_2)$.
Since, we get $|G| = |T||S|$ and for some $s \in S,$ in G each right coset of $im(\mu)$ may be written as $T\xi(s)$. Thus there is a unique representation of all elements $g \in G$ in terms of $\mu(t)\xi(s)$ for $t \in T$ and $s \in S$. Particularly, $s = \eta(g)$ and $t = \omega(g\xi(s^{-1}))$. As

$$
\begin{aligned}
&\mu(t)\xi(s) = \mu\left(\omega(g\xi(s^{-1}))\right)\xi(s) \\
&= \mu \circ \omega(g\xi(s^{-1}))\xi(s) \\
&= g\xi(s^{-1})\xi(s) \\
&= g
\end{aligned}
$$

Now define $\Theta : G \to T \rtimes_\tau S$ by $\Theta(g) = ts$ for $t \in T$ and $s \in S$. We have to prove that Θ is 1-1 onto. Without loss of generality, let $g = \mu(t)\xi(s)$ and $g_1 = \mu(t_1)\xi(s_1)$.
Suppose $g, g_1 \in G$ so that

$$
\begin{aligned}
&\Theta(g) = \Theta(g_1) \\
&\Rightarrow ts = t_1 s_1 \\
&\Rightarrow t = t_1 \ \&\ s = s_1
\end{aligned}
$$

And hence

$$\mu(t) = \mu(t_1) \ \&\ \xi(s) = \xi(s_1).$$

So,

$$
\begin{aligned}
&\mu(t)\xi(s) = \mu(t_1)\xi(s_1) \\
&\Rightarrow g = g_1 \\
&\Rightarrow \Theta \text{ is 1-1.}
\end{aligned}
$$

Clearly Θ is onto. To show Θ is isomorphism it is sufficient to show that Θ is homomorphism that is $\Theta(gg_1) = \Theta(g)\Theta(g_1)\ \forall\ g, g_1 \in G$.

Now

$$\begin{aligned}
\Theta(gg_1) &= \Theta\big(\mu(t)\xi(s)\mu(t_1)\xi(s_1)\big) \\
&= \Theta\big(\mu(t)\xi(s)\mu(t_1)\xi(s^{-1})\xi(s)\xi(s_1)\big) \\
&= \Theta\big(\mu(t)\xi(s)\mu(t_1)\xi(s^{-1})\xi(ss_1)\big) \\
&= \Theta\left(\mu(t)id_{im(\mu)}\big(\xi(s)\mu(t_1)\xi(s^{-1})\xi(ss_1)\big)\right) \\
&= \Theta\left(\mu(t)(\mu \circ \omega)\big(\xi(s)\mu(t_1)\xi(s^{-1})\xi(ss_1)\big)\right) \\
&= \Theta\left(\ \mu(t)\mu\big[\omega\big(\xi(s)\mu(t_1)\xi(s^{-1})\big)\big]\xi(ss_1)\right) \\
&= \Theta\left(\mu(t)\mu\big(\tau_s(t_1)\xi(ss_1)\big)\right) \\
&= \Theta\left(\mu(t\tau_s(t_1))\xi(ss_1)\right) \\
&= t\tau_s(t_1)ss_1 \\
&= tst_1s^{-1}ss_1 \\
&= tst_1s_1 \\
&= \Theta\big(\mu(t)\xi(s)\big)\Theta\big(\mu(t_1)\xi(s_1)\big) \\
&= \Theta(g)\Theta(g_1)
\end{aligned}$$

Hence, $\Theta : G \to T \rtimes_\tau S$ defined by $\Theta(g) = ts$ is isomorphism. So $G \cong T \rtimes_\tau S$.

Theorem: Assume that $p_1, p_2, \dots, p_s$ are distinct primes such that $n = p_1^{t_1}p_2^{t_2}\dots p_s^{t_s}$ be a $+ve$ integer, where $t_1, t_2, \dots, t_s \in \mathbb{N}$. Then
$Z_n \cong Z_{p_1^{t_1}} \times Z_{p_2^{t_2}} \times \dots \times Z_{p_s^{t_s}}$.

Proof: Let $G = Z_{p_1^{t_1}} \times Z_{p_2^{t_2}} \times \dots \times Z_{p_s^{t_s}}$ then $|G| = p_1^{t_1}p_2^{t_2}\dots p_s^{t_s}$, we shall prove $G \cong Z_n$. For that we shall show that G is generated by a single element of order n. Suppose the set of generators $\{z_1, z_2, \dots, z_s\}$ of G so that $Z_{p_i^{t_i}}$, a subgroup of G, is generated by z_i. $|z_1z_2\dots z_s|$ in G is least $+ve$ integer m so that $(z_1z_2\dots z_s)^m = e$. But $(z_1z_2\dots z_s)^m = e \ \ iff \ \ z_i^m = e \ \forall \ 1 \le i \le s$.

So, we have

$$(z_1z_2\dots z_s)^m = e \ \ iff \ \ m \equiv 0\big(mod\ p_i^{t_i}\big) \ \forall \ 1 \le i \le s.$$

Since

$$\big(p_i^{t_i}, p_j^{t_j}\big) = 1 \ \ if\ i \ne j,$$

Therefore, m satisfying

$$m \equiv 0\big(mod\ p_i^{t_i}\big) \ \forall \ 1 \le i \le s,$$

Is $p_1^{t_1}p_2^{t_2}\dots p_s^{t_s}$, and Hence $|z_1z_2\dots z_s| = n$. So $G \cong Z_n$ or $Z_n \cong Z_{p_1^{t_1}} \times Z_{p_2^{t_2}} \times \dots \times Z_{p_s^{t_s}}$.

Hence the result.

Theorem: If $G \cong Z_n$, in that case $Aut(G) \cong Z_n^*$, where $Z_n^* -$ multiplicative group of units in Z_n.

Proof: Assume that $Z_n = < s >$. Then $o(s) = n$. Suppose $\mu \in Aut(G)$. Since μ is onto so $\forall\ t \in G\ \exists\ r \in G$ so that $\mu(r) = t$. Again, the group G is cyclic so $r = s^x$ for $x \in \mathbb{Z}$.

Thus, we get

$$\begin{aligned}
t &= \mu(r) \\
&= \mu(s^x) \\
&= (\mu(s))^x
\end{aligned}$$

$\Rightarrow \mu(s)$ is also the generator of G.

As there are only $\phi(n)$, Eular ϕ-function, generators of G which implies there are only $\phi(n)$ choices for μ.
Hence $|Aut(G)| \le \phi(n)$.
Define $\mu_h : G \to G$ by

$\mu_h(k) = k^h, 1 \leq h \leq n \;\&\; (h,n) = 1$

Then for $k_1, k_2 \in G$, we get

$$\mu_h(k_1k_2) = (k_1k_2)^h$$
$$= (k_1)^h(k_2)^h$$
$$= \mu_h(k_1)\mu_h(k_2)$$

$\Rightarrow \mu_h$ is homomorphism.

Suppose $g_1, g_2 \in G$ so that $g_1 \neq g_2$. Let $g_1 = s^c$ and $g_2 = s^d$. Then

$\mu_h(g_1) = g_1^h$

$= (s^c)^h$

$= s^{ch}$

And

$\mu_h(g_2) = g_2^h$

$= (s^d)^h$

$= s^{dh}$

$\Rightarrow \mu_h(g_1) - \mu_h(g_2)s^{(c-d)h}$

$\Rightarrow \mu_h(g_1) - \mu_h(g_2) \;\; iff \;\; s^{(c-d)h} = e$

Since $(h,n) = 1$. So $s^{(c-d)h} = e$ if $n/(c-d)$ or $c - d = nw$ for some $w \in \mathbb{Z}$.

Now if $c - d = nw$, in that case we have

$$g_2 = s^d$$
$$= s^d e$$
$$= s^d s^{nw}$$
$$= s^d s^{c-d}$$
$$= s^c = g_1.$$

However $g_1 \neq g_2$, so $n \nmid (c-d)$ and hence

$\mu_h(g_1) \neq \mu_h(g_2)$

$\Rightarrow \mu_h$ is 1-1.

For $g_2 \in G \; \exists \; g_1 \in G$ so that $g_2 = g_1^h$, therefore

$g_2 = g_1^h = \mu_h(g_1)$

$\Rightarrow \mu_h$ is onto.

$\Rightarrow \mu_h \in Aut(G)$

If

$\mu_u = \mu_v$ then $\mu_u(s) = \mu_v(s)$

$\Rightarrow s^u = s^v$.

Let $u > v$ then $s^{u-v} = e$

$\Rightarrow o(s)/u - v$

$\Rightarrow n \leq u - v \leq n.$

It is a contradiction. Hence $\mu_u \neq \mu_v \; \forall \; u, v \; (u \neq v), 1 \leq u, v \leq n$ and $(u,n) = 1 = (v,n)$.

$\Rightarrow$ the minimum number of automorphism of G are $\phi(n)$.

$$\Rightarrow |Aut(G)| = \phi(n).$$

Or

$$Aut(G) = \mu_h;\ \mu_h(k) = k^h, 1 \le h \le n \ \&\ (h,n) = 1$$

Define $\Phi : Aut(G) \to Z_n^*$ so that

$$\Phi(\mu_h) = h, 1 \le h < n \ \&\ (h,n) = 1.$$

For $h \in Z_n^*, 1 \le h < n, (h,n) = 1\ \exists\ \mu_h \in Aut(G)$ so that $\Phi(\mu_h) = h$.

$\Rightarrow \Phi$ is onto.

Let $\mu_h, \mu_k \in Aut(G)$ so that

$$\Phi(\mu_h) = \Phi(\mu_k)$$
$$\Rightarrow h = k$$
$$\Rightarrow \mu_h = \mu_k$$
$\Rightarrow \Phi$ is 1-1.

Again, for $\mu_h, \mu_k \in Aut(G)$, we get

$$\mu_h \circ \mu_k(g) = \mu_h(\mu_k(g))$$
$$= \mu_h(g^k)$$
$$= (g^k)^h$$
$$= g^{kh}$$
$$= \mu_{hk}(g)$$
$$\Rightarrow \Phi(\mu_h \circ \mu_k) = \Phi(\mu_{hk})$$
$$= hk$$
$$= \Phi(\mu_h)\Phi(\mu_k)$$

$\Rightarrow \Phi -$ Homomorphism.
$\Rightarrow \Phi$ Is automorphism.

$\Rightarrow Aut(G) \cong Z_n^*$

Theorem: Suppose $\boldsymbol{G \cong Z_n \times Z_2}$ for $\boldsymbol{+ve}$ integer $\boldsymbol{n}$, then

$$|\boldsymbol{Aut(G)}| = \begin{cases} \boldsymbol{\phi(n)}; & \boldsymbol{n - odd} \\ \boldsymbol{6\phi(n)}; & \boldsymbol{n \equiv 2(mod\ 4)} \\ \boldsymbol{4\phi(n)}; & \boldsymbol{n \equiv 0(mod\ 4)} \end{cases}.$$

Proof: As $G \cong Z_n \times Z_2$, so that $o(G) = 2n$.

1. For odd n, in that case $(n,2) = 1$ so $Z_n \times Z_2 \cong Z_{2n}$.

And $|Aut(Z_{2n}) = \phi(n)$.

2. For even n, Let $Z_n =< c >\ \ \&\ Z_2 =< d >$ such that

$$Z_n \times Z_2 =< c > \times < d >.$$

Thus for $\mu \in Aut(Z_n \times Z_2)$ and $o(c) = n$

$$\Rightarrow o(\mu(c)) = n$$

And

$o(d) = 2 \Rightarrow o(\mu(d)) = 2$

And

$\mu^{\left(\frac{n}{2}\right)}(c) \neq \mu(d),$

Where $c, d \in Z_n \times Z_2$.

Again $\mu(c)$ & $\mu(d)$ generates $Z_n \times Z_2 \ iff \ < \mu(c) > \cap < \mu(d) > = \{e\}$.

Thus, for finding $|Aut(Z_n \times Z_2)|$, we find out the number of pairs (s_1, s_2) which satisfies

a. $s_1 \in Z_n \times Z_2$ so that $o(s_1) = n$
b. $s_2 \in Z_n \times Z_2$ so that $o(s_2) = 2$
c. $s_1^{\left(\frac{n}{2}\right)} \neq s_2$.

Case 1. Suppose $n \equiv 2(mod\ 4)$. As $o(c^j)$ and $o(c^j d)$ is n whenever $(j, n) = 1$.
The possible all generators are not there for Z_n, as $n \equiv 2(mod\ 4)$

$\Rightarrow n - 2 = 4l,$

Where $l \in \mathbb{Z}$.

$\Rightarrow n = 2 + 4l$

Or

$\frac{n}{2} = 1 + 2l$

$\Rightarrow \frac{n}{2} - odd$

$o(c^j d) = n$ whenever $o(c^j) = \frac{n}{2}$.

$\Rightarrow$There exists $\phi\left(\frac{n}{2}\right)$ elements whose order are $\frac{n}{2}$ of form c^j .

Or n order $\phi\left(\frac{n}{2}\right) + 2\phi(n)$ elements exists in $Z_n \times Z_2$.

Again $\phi\left(\frac{n}{2}\right) = \phi(n)$

$\Rightarrow$ There are $3\phi(n)$ elements belongs to $Z_n \times Z_2$ which has order n and 2 order elements are $d,\ c^{\left(\frac{n}{2}\right)}d,\ c^{\left(\frac{n}{2}\right)} \in Z_n \times Z_2$.

$\forall\ s_1 \in Z_n \times Z_2$ so that $o(s_1) = n\ \ \exists\ 2$ elements satisfying $s_1^{\left(\frac{n}{2}\right)} \neq s_2$.

So, $|Aut(Z_n \times Z_2)| = 6\phi(n)$.

Case 2 Suppose $n \equiv 0(mod\ 4)$. So for $s_1 \in Z_n \times Z_2$, $\quad o(s_1) = n\ iff$
Either

$s_1 = c^j\ \ \&\ \ (j,n) = 1$

Or

$s_1 = c^j d\ \ \&\ \ (j,n) = 1.$

For $s_1 \in Z_n \times Z_2$ having order n, then those elements which satisfies

$s_1^{\left(\frac{n}{2}\right)} \neq s_2$

Are d as well as $c^{\left(\frac{n}{2}\right)}d$ and those have order 2.

Also

$$\left(c^{\left(\frac{n}{2}\right)}\right)^2 = c^n = e$$

$$\Rightarrow o\left(c^{\left(\frac{n}{2}\right)}\right) = 2.$$

As

$$n \equiv 0(mod\ 4)$$
$$\Rightarrow s_1^{\left(\frac{n}{2}\right)} = c^{\left(\frac{n}{2}\right)}$$

For all n order elements.

Thus there exists 2 elements from $Aut(Z_n \times Z_2)$ which has $2\phi(n)$ elements having order n, that sends $c \mapsto s_1$ and $d \mapsto d$ or $c \mapsto s_1$ and $d \mapsto c^{\left(\frac{n}{2}\right)}d$.

Thus $|Aut(Z_n \times Z_2)| = 4\phi(n)$.

Theorem: $Aut(Z_n \times Z_2)$ is isomorphic to $(Aut(Z_n) \times Z_2) \rtimes Z_2$, where $n = 4k$ is $+ve$ integer and $k \in \mathbb{Z}$.

Proof: Assume $Z_n = <c> \& Z_2 = <d>$ then $Z_n \times Z_2 = <c> \times <d>$. As $n \equiv 0(mod\ 4)$.

Let $\xi_{(s_1,s_2)} \in Aut(Z_n \times Z_2)$ such that

$$\xi_{(s_1,s_2)} : \begin{cases} c \to s_1 \\ d \to s_2 \end{cases}$$

Where $s_1, s_2 \in Z_n \times Z_2$.

Suppose $T \subseteq Aut(Z_n \times Z_2)$ be a subgroup such that $\forall \rho \in T \Rightarrow \rho(d) = d$.

Now we will prove that $T \cong Aut(Z_n) \times Z_2$ as well as $Aut(Z_n \times Z_2) \cong T \rtimes Z_2$. Any $\rho \in T$ are either $\xi_{(c^i,d)}$ or $\xi_{(c^i d,d)}$ form so that $(i, n) = 1$.

The remaining elements of $Aut(Z_n \times Z_2)$ other than T are of form $\xi_{\left(c^i,c^{\left(\frac{n}{2}\right)}d\right)}$ or $\xi_{\left(c^i d,c^{\left(\frac{n}{2}\right)}d\right)}$ such that $(i, n) = 1$. Then

$[Aut(Z_n \times Z_2) : T] = 2$.

This implies $T \trianglerighteq Aut(Z_n \times Z_2)$. Given the natural surjection $\mu : Aut(Z_n \times Z_2) \to Z_2$ and injection $\nu : T \to Aut(Z_n \times Z_2)$ which sends elements belongs to T to e, where e is unit element of Z_2. Thus we get an exact short sequence

$$\{e\} \to T \xrightarrow{\nu} Aut(Z_n \times Z_2) \xrightarrow{\mu} Z_2 \to \{e\}.$$

Let $\theta : Z_2 \to Aut(Z_n \times Z_2)$ which sends $e \neq z_2 \in Z_2$ to $\xi_{\left(c,c^{\left(\frac{n}{2}\right)}d\right)} \in Aut(Z_n \times Z_2)$ and $e \in Z_2$ to some elements of T.

Now,

$$\mu \circ \theta(1) = \mu(\theta(1))$$
$$= \mu\left(c, c^{\left(\frac{n}{2}\right)}d\right)$$
$$= 1$$

And

$$\mu \circ \theta(0) = \mu(\theta(0))$$
$$= \mu(t)$$
$$= 0$$

Where $t \in T$.

$\Rightarrow \mu \circ \theta = id_{Z_2}$.

Thus θ splits exact short sequence

$$\{e\} \to T \xrightarrow{\nu} Aut(Z_n \times Z_2) \xrightarrow{\mu} Z_2 \to \{e\}.$$

Then

$$Aut(Z_n \times Z_2) \cong T \rtimes Z_2.$$

Now, we shall prove that $T \cong Aut(Z_n) \times Z_2$.
Assume an exact short sequence $\{e\} \to Aut(Z_n) \xrightarrow{\gamma} T \xrightarrow{\delta} Z_2 \to \{e\}$

so that γ maps $\{c \mapsto c^j\}$ to $\xi_{(c^j,d)}$ as well as δ maps $\xi_{(c^j d,d)} \to$ non-unit element of Z_2. There exists $\theta : Z_2 \to T$ which sends non-unit element to $\xi_{(cd,d)}$. Thus $T \cong Aut(Z_n) \rtimes Z_2$.
As

$$n \equiv 0(mod\ 4), \qquad (i,n) = 1$$
$$\Rightarrow i - odd.$$

For any j,

$$\xi_{(c^i,d)} \circ \xi_{(cd,d)}(c^j) = \xi_{(c^i,d)}\left(\xi_{(cd,d)}(c^j)\right)$$
$$= \xi_{(c^i,d)}(c^j d^j)$$
$$= c^{ij} d^j$$

And

$$\xi_{(cd,d)} \circ \xi_{(c^i,d)}(c^j) = \xi_{(cd,d)}\left(\xi_{(c^i,d)}(c^j)\right)$$
$$= \xi_{(cd,d)}(c^{ij})$$
$$= c^{ij} d^{ij}$$

As $d \in Z_2\ \&\ i - odd$, therefore for even j, $d^j = e = d^{ij}$ as well as for odd j, $d^j = d = d^{ij}$. Thus we get

$$\xi_{(c^i,d)} \circ \xi_{(cd,d)}(c^j) = \xi_{(cd,d)} \circ \xi_{(c^i,d)}(c^j)$$

On the same we have to prove that

$$\xi_{(c^i,d)} \circ \xi_{(cd,d)}(c^j d) = \xi_{(cd,d)} \circ \xi_{(c^i,d)}(c^j d)$$
$$\Rightarrow \xi_{(c^i,d)} \circ \xi_{(cd,d)} = \xi_{(cd,d)} \circ \xi_{(c^i,d)}\ \forall\, i, (i,n) = 1.$$
$$\Rightarrow T \cong Aut(Z_n) \times Z_2$$

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