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On the hyperbola $y^2 = 8x^2 + 16$

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Abstract
The binary quadratic equation represented by the positive pellian $y^2 = 8x^2 + 16$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: binary quadratic, hyperbola, parabola, integral solutions, pell equation

1. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ Where D is non-square positive integer has been studied by various mathematics for its non-trivial integral solutions when D takes different integral value [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 8x^2 + 16$ is considered and infinity many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. Method of Analysis

The positive pell equation representing hyperbola under consideration is,

$$y^2 = 8x^2 + 16 \tag{1}$$

The smallest positive integer solutions of (1) is $x_0 = 4, y_0 = 12$

The general solutions $(x_{n,}, y_n)$ of (1) is given by

$$y_n = \frac{1}{2} f_n, x_n = \frac{1}{2\sqrt{8}} g_n \tag{2}$$

Where,

$$f_n = (3+\sqrt{8})^{n+1} + (3-\sqrt{8})^{n+1}$$
$$g_n = (3+\sqrt{8})^{n+1} - (3-\sqrt{8})^{n+1}$$

The recurrence relation satisfied by the solutions (2) are given by

$$6y_{n+2} - y_{n+3} - y_{n+1} = 0$$
$$6x_{n+2} - x_{n+3} - x_{n+1} = 0$$

Some numerical examples of x & y satisfying (1) are given in the table below

Table 1: Examples

N	x_n	y_n
0	4	12
1	24	68
2	140	396
3	816	2308
4	4756	13452

From the above table, we observe some interesting relation among the solutions which are presented below:

1. x_n & y_n values are even.
2. Each of the following expression is a nasty numbers:

- $[9x_{2n+3}-51x_{2n+2}+12]$
- $\frac{[114x_{2n+2}-3x_{2n+4}+24]}{2}$
- $3y_{2n+3}-16x_{2n+2}+12$
- $\frac{[18y_{2n+4}-1680x_{2n+2}+408]}{34}$
- $\frac{[102x_{2n+4}-594x_{2n+3}+24]}{2}$
- $\frac{[34y_{2n+2}-16x_{2n+3}+24]}{2}$
- $\frac{[102y_{2n+3}-288x_{2n+3}+24]}{2}$
- $\frac{[560x_{2n+3}-34y_{2n+4}+24]}{2}$
- $\frac{[594y_{2n+2}-48x_{2n+4}+408]}{34}$
- $\frac{[96x_{2n+4}-198y_{2n+3}+24]}{2}$
- $\frac{[594y_{2n+4}-1680x_{2n+4}+24]}{2}$
- $[18y_{2n+2}-3y_{2n+3}+12]$
- $\frac{[1680y_{2n+3}-288y_{2n+4}+192]}{16}$
- $[35y_{2n+2}-y_{2n+4}+24]$

3. Each of the following expression is a cubical integer:

- $12(3x_{n+2}-17x_{n+1})+4(3x_{3n+4}-17x_{3n+3})$
- $16(38x_{3n+3}-x_{3n+5})+48(38x_{n+1}-x_{n+3})$
- $4(y_{3n+4}-16x_{3n+3})+12(y_{n+2}-16x_{n+1})$
- $4624(6y_{3n+5}-560x_{3n+3})+13872(6y_{n+3}-560x_{n+1})$
- $2(34x_{3n+5}-198x_{3n+4})+6(34x_{n+3}-198x_{n+2})$
- $144(34y_{3n+3}-16x_{3n+4})+432(34y_{n+1}-16x_{n+2})$
- $2(34y_{3n+4}-96x_{3n+4})+6(34y_{n+2}-96x_{n+2})$
- $144(34y_{3n+5}-560x_{3n+4})+432(34y_{n+3}-560x_{n+2})$
- $4624(198y_{3n+3}-16x_{3n+5})+13872(198y_{n+1}-16x_{n+3})$
- $144(198y_{3n+4}-96x_{3n+5})+432(198y_{n+2}-96x_{n+3})$
- $4(198y_{3n+5}-560x_{3n+5})+12(198y_{n+3}-560x_{n+3})$
- $144(35y_{3n+3}-y_{3n+5})+432(35y_{n+1}-y_{n+3})$
- $4(6y_{3n+3}-y_{3n+4})+12(6y_{n+1}-y_{n+2})$
- $1024(560y_{3n+4}-96y_{3n+5})+3072(560y_{n+2}-96y_{n+3})$

4. Relations among the solutions:

- $2y_{n+1} = 1086x_{n+1} - 180x_{n+2}$
- $2y_{n+2} = 6x_{n+2} - 2x_{n+1}$
- $2y_{n+3} = 34x_{n+2} - 6x_{n+1}$
- $12y_{n+1} = 2x_{n+3} - 34x_{n+1}$
- $2y_{n+2} = x_{n+3} - x_{n+1}$
- $12y_{n+3} = 34x_{n+3} - 2x_{n+1}$
- $2x_{n+3} = 4y_{n+2} + 2x_{n+1}$
- $6y_{n+1} = 2y_{n+2} - 16x_{n+1}$
- $6y_{n+3} = 34y_{n+2} + 16x_{n+1}$
- $136x_{n+2} = 8y_{n+3} - 12760x_{n+1}$
- $136x_{n+3} = 48y_{n+3} + 8x_{n+1}$
- $136y_{n+1} = 8y_{n+3} - 384x_{n+1}$
- $2y_{n+1} = 6x_{n+3} - 34x_{n+2}$
- $2y_{n+2} = 2x_{n+3} - 6x_{n+2}$
- $2y_{n+3} = 6x_{n+3} - 2x_{n+2}$
- $24x_{n+1} = 8x_{n+2} - 8y_{n+1}$
- $24x_{n+3} = 8y_{n+1} + 136x_{n+2}$
- $2y_{n+3} = 2y_{n+1} - 32x_{n+2}$
- $2y_{n+1} = 6y_{n+2} - 16x_{n+2}$
- $2y_{n+3} = 6y_{n+2} + 16x_{n+2}$
- $2x_{n+1} = 34x_{n+2} - 2y_{n+3}$
- $24x_{n+3} = 8y_{n+3} + 8x_{n+2}$
- $24y_{n+2} = 8y_{n+3} - 64x_{n+2}$
- $34y_{n+2} = 6y_{n+1} + 16x_{n+3}$
- $136y_{n+3} = 8y_{n+1} + 384x_{n+3}$
- $6x_{n+1} = 6x_{n+3} - 12y_{n+2}$
- $12x_{n+2} = 4x_{n+3} - 4y_{n+2}$
- $12y_{n+1} = 68y_{n+2} - 32x_{n+3}$
- $12y_{n+3} = 4y_{n+2} + 32x_{n+3}$
- $2y_{n+1} = 34y_{n+3} - 96x_{n+3}$
- $2y_{n+2} = 6y_{n+3} - 16x_{n+3}$
- $32x_{n+1} = 4y_{n+2} - 12y_{n+1}$
- $32x_{n+2} = 12y_{n+2} - 4y_{n+1}$
- $2y_{n+3} = 12y_{n+2} - 2y_{n+1}$
- $96x_{n+1} = 2y_{n+3} - 34y_{n+1}$
- $24y_{n+2} = 4y_{n+1} + 4y_{n+3}$
- $32x_{n+1} = 8y_{n+3} - 68y_{n+2}$
- $64x_{n+3} = 24y_{n+3} - 8y_{n+2}$

- $64y_{n+1} = 384y_{n+2} - 64y_{n+3}$
- $24y_{n+2} = 8y_{n+1} + 64x_{n+2}$

Remarkable Observation

i) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table 2 below.

Table 2: Hyperbolas

S. No.	(Xn,Yn)	Hyperbola
1.	$(6x_{n+1} - x_{n+2}, 3x_{n+2} - 17x_{n+1})$	$Y_n^2 - 8X_n^2 = 16$
2.	$(35x_{n+1} - x_{n+3}, x_{n+3} - 33x_{n+1})$	$18Y_n^2 - 16X_n^2 = 1152$
3.	$(17x_{n+1} - y_{n+2}, y_{n+2} - 16x_{n+1})$	$9Y_n^2 - 8X_n^2 = 144$
4.	$(99x_{n+1} - y_{n+3}, 6y_{n+3} - 560x_{n+1})$	$Y_n^2 - 32X_n^2 = 18496$
5.	$(70x_{n+2} - 12x_{n+3}, 34x_{n+3} - 198x_{n+2})$	$Y_n^2 - 8X_n^2 = 64$
6.	$(x_{n+2} - 2y_{n+1}, 34y_{n+1} - 16x_{n+2})$	$Y_n^2 - 288X_n^2 = 576$
7.	$(34x_{n+2} - 12y_{n+2}, 34y_{n+2} - 96x_{n+2})$	$Y_n^2 - 8X_n^2 = 64$
8.	$(33x_{n+2} - 2y_{n+3}, 34y_{n+3} - 560x_{n+2})$	$Y_n^2 - 288X_n^2 = 576$
9.	$(3x_{n+3} - 35y_{n+1}, 198y_{n+1} - 16x_{n+3})$	$Y_n^2 - 32X_n^2 = 18496$
10.	$(34x_{n+3} - 70y_{n+2}, 198y_{n+2} - 96x_{n+3})$	$Y_n^2 - 8X_n^2 = 576$
11.	$(y_{n+3} - 33y_{n+1}, 35y_{n+1} - y_{n+3})$	$256Y_n^2 - 288X_n^2 = 147456$
12.	$(6y_{n+2} - 34y_{n+1}, 6y_{n+1} - y_{n+2})$	$64Y_n^2 - 2X_n^2 = 1024$
13.	$(198x_{n+3} - 70y_{n+3}, 198y_{n+3} - 560x_{n+3})$	$Y_n^2 - 8X_n^2 = 64$
14	$(34y_{n+3} - 198y_{n+2}, 560y_{n+2} - 96y_{n+3})$	$Y_n^2 - 8X_n^2 = 4096$

ii) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of_parabolas which are presented in the table 3 below:

Table 3: Parabolas

S. No.	(Xn,Yn)	Parabolas
1.	$(6x_{n+1} - x_{n+2}, 3x_{2n+3} - 17x_{2n+2})$	$Y_n - 4X_n^2 = 4$
2.	$(35x_{n+1} - x_{n+3}, 38x_{2n+2} - x_{2n+4})$	$18Y_n - 4X_n^2 = 144$
3.	$(17x_{n+1} - y_{n+2}, y_{2n+3} - 16x_{2n+2})$	$9Y_n - 4X_n^2 = 54$
4.	$(99x_{n+1} - y_{n+3}, 6y_{2n+4} - 560x_{2n+2})$	$289Y_n - 136X_n^2 = 39304$
5.	$(70x_{n+2} - 12x_{n+3}, 34x_{2n+4} - 198x_{2n+3})$	$Y_n - 2X_n^2 = 8$
6.	$(x_{n+2} - 2y_{n+1}, 34y_{2n+2} - 16x_{2n+3})$	$Y_n - 24X_n^2 = 24$
7.	$(34x_{n+2} - 12y_{n+2}, 34y_{2n+3} - 96x_{2n+3})$	$Y_n - 2X_n^2 = 8$
8.	$(33x_{n+2} - 2y_{n+3}, 560x_{2n+3} - 34y_{2n+4})$	$Y_n - 24X_n^2 = 24$
9.	$(3x_{n+3} - 35y_{n+1}, 198y_{2n+2} - 16x_{2n+4})$	$289Y_n - 136X_n^2 = 39304$
10.	$(34x_{n+3} - 70y_{n+2}, 96x_{2n+4} - 198y_{2n+3})$	$18Y_n - 12X_n^2 = 432$
11.	$(198x_{n+3} - 70y_{n+3}, 198y_{2n+4} - 560x_{2n+4})$	$Y_n - 2X_n^2 = 8$
12.	$(6y_{n+2} - 34y_{n+1}, 6y_{2n+2} - y_{2n+3})$	$128Y_n - 2X_n^2 = 512$
13.	$(y_{n+3} - 33y_{n+1}, 35y_{2n+2} - y_{2n+4})$	$256Y_n - 24X_n^2 = 6144$
14	$(34y_{n+3} - 198y_{n+2}, 560y_{2n+3} - 96y_{2n+4})$	$8Y_n - 2X_n^2 = 512$

iii) Consider , $m = x_{n+1} + y_{n+1}, n = x_{n+1}$ .observe that $m > n > 0$ .Treat m,n as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where $\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2$

Then the following interesting relation are observed.

a) $\alpha - 4\beta + 3\gamma = -16$

b) $\frac{2A}{P} = x_{n+1} y_{n+1}$

c) $\gamma - 5\alpha = -\frac{16A}{P} + 16$

d) $3\alpha - 2\beta + \gamma = \frac{8A}{P} - 16$

Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 8x^2 + 16$. As the binary quadratic Diphontine equation are rich in variety, one may search for the other choices of positivepell equations and determine their integer solutions along with suitable properties.

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