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Common fixed point theorem for a particular family of mappings in the fuzzy metric space

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Abstract

In this paper, a common fixed point theorem for a particular family of mappings of a G-Complete fuzzy metric space (X, M, ∗) into itself is proved. From this result, a common fixed point theorem for a family of mappings of a complete metric space (X, d) into itself is also proved.

Keywords: φ −weak contraction, common fixed point.

1. Introduction

The notion of fuzzy metric space [1, 2-4, 6, 9, 10] were introduced and the Banach fixed point theorem to contractive mappings on complete fuzzy metric were introduced. It is proved that a common fixed point result for a class of mappings in the fuzzy metric spaces in the sense of Kramosil and Michalek [9] which are complete in Grabiec's sense [5]. The result is proved by modifying the contractility definition given by Gregori and Sapena [7]. The result is also allowed to get a common fixed point result for a family of mappings of a complete metric space (X, d) into itself.

2. Preliminaries on the fuzzy metric spaces

In this section, some notions and some results on the fuzzy metric spaces are given.

Definition 1. (Schweizer and Sklar [10]). A binary operation ∗: [0, 1]×[0, 1]→[0, 1] is a continuous t-norm if it satisfies the following conditions:

i) ∗ is associative and commutative,
ii) ∗ is continuous,
iii) $a * 1 = a$ for every $a \in [0, 1]$,
iv) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2. (Kramosil and Michalek [9]). A triplet (X, M, ∗) is a fuzzy metric space if X is an arbitrary set, ∗ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, +\infty)$ satisfying, for every $x, y, z \in X$, the following conditions:

i) $M(x, y, 0) = 0$,
ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
iii) $M(x, y, t) = M(y, x, t)$,
iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
v) $M(x, y, \cdot):[0, +\infty) \to [0, 1]$ is left-continuous.

Remark 1. It is easy to prove the M (x, y, ·) is non-decreasing for every $x, y \in X$.

If (X, M, ∗) is a fuzzy metric space we can say that M is a fuzzy metric on X. Let (X, d) be a metric space. Let $a * b = ab$ for every $a, b \in [0, 1]$ and let $M_d : X \times X \times [0, +\infty) \to [0, 1]$ be the function defined, for all $x, y \in X$, by $M_d(x, y, 0) = 0$ and for $t > 0$ by

$$M_d(x, y, t) = \frac{t}{t + d(x,y)}.$$

The triplet (X, M_d, ∗) is a fuzzy metric space and M_d is called the fuzzy metric induced by d.

Definition 3. (Grabiec [5]). A sequence (x_n) in a fuzzy metric space (X, M, ∗) is G-Cauchy if $\lim_{n\to+\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for every $p > 0$, A sequence (x_n) in a fuzzy metric space (X, M, ∗) converges to $x \in X$ if $M(x_n, x, t) \to 1$, as $n \to +\infty$, for every $t > 0$.

By Corollary 7 in [5], we obtain the result.

Lemma 1. Let (X, M, ∗) be a fuzzy metric space and let $(x_n) \subset X$. If $x_n \to x \in X$ and if M (x, y, ·) is right-continuous for $t > 0$, then

$\lim_{n\to\infty} M(x_n, y, t) = M(x, y, t)$, for every $t > 0$.

Definition 4. ([7] Definition 3.5), Let (X, M, ∗) be a fuzzy metric space and let $T : X \to X$. The map T is a fuzzy contraction if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for every $x, y \in X$ and for every $t > 0$.

Gregori and Sapena [7] is proved for fuzzy contractive mapping the contraction theorem of Banach, precisely:

Theorem 1. ([7] Theorem 5.2). Let (X, M, ∗) be a G-complete fuzzy metric space and let $T : X \to X$ be a contractive mapping. Then T has a unique fixed point.

In a fuzzy metric space (X, M, ∗) a sequence (x_n) is called fuzzy contractive ([7] Definition 3.8) if there exists $0 < k < 1$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)$$

for every $n \in \mathbb{N}$.

Definition 5. Let (X, M, ∗) be a fuzzy metric space. The fuzzy set M (x, y, t) is regular if for every fuzzy contractive sequence (x_n) that converges to $x \in X$

$\lim_{n\to+\infty} M(x_n, y, t) = M(x, y, t)$,

holds, for every $y \in X$ and for every $t > 0$.

In virtue of Lemma I, we have the following result.

Lemma 2. Let (X, M, ∗) be a fuzzy metric space. If M (x, y, ·) is right continuous for every $t > 0$, then M(x, y, t) is regular.

Remark 2. Any fuzzy metric M_d induced by a metric d is regular.

Theorem 2. ([12] Theorem 2) Let (X, M, ∗) be a G-Complete fuzzy metric space and let F be a family of mappings of X into itself. We suppose that:

i) M(X, M, t) is regular.

ii) there exists T ∈ F such that

$$\frac{1}{M\,(Tx,Sy,t)} - 1 \leq a \left(\frac{1}{M\,(x,y,t)} - 1\right) + b\left(\frac{1}{M\,(x,Tx,t)} - 1\right) + c\left(\frac{1}{M\,(y,Sy,t)} - 1\right)$$

for every S∈ F, for every x ∈X and y∈ T(X) with a, b, c ∈ [0, 1) such that a + b +c <1.

Then there exists a unique fixed point that is common to all the mapping of F.

3. The Main Results

Theorem 3. Let (X, M, ∗) be a G-Complete fuzzy metric space and let F be a family of mappings of X into itself. We suppose that:

1. M(X, M, t) is regular
2. there exists T ∈ F such that

$$\frac{1}{M\,(Tx,Sy,t)} - 1 \leq a \max\left\{\frac{1}{M\,(x,y,t)} - 1, \frac{1}{M\,(x,Tx,t)} - 1, \frac{1}{M\,(y,Sy,t)} - 1\right\} \quad (1)$$

for every S∈F, for every x ∈X and y∈T(X) with a ∈ [0, 1).

Then there exists a unique fixed point that is common to all the mapping of F.

Proof Fix x∈X and Consider that sequence <x_n> in X with $x_n = T^n(x)$

The (1), for S = T, implies

$$\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 \leq a \max\left\{\frac{1}{M(x_n,x_{n+1},t)} - 1, \frac{1}{M(x_n,x_{n+1},t)} - 1, \frac{1}{M(x_{n+1},x_{n+2},t)} - 1\right\}$$

If $\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 \leq a\left\{\frac{1}{M(x_{n+1},x_{n+2},t)} - 1\right\}$

$\Rightarrow M(x_{n+1},x_{n+2},t) = 1$ for every t > 0. $\forall\, n \in$ N.

Again if

$$\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 \leq a\left(\frac{1}{M(x_n,x_{n+1},t)} - 1\right)$$

So the sequence <x_n> is contractive.

By induction, it follows that

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \leq a^n\left(\frac{1}{M(x,x_1,t)} - 1\right) \to 0 \text{ as } n \to +\infty$$

Consequently $M(x_n,x_{n+1},t) \to 1$ for every t > 0.

For a fixed P ∈ N, Consider

$$M(x_n,x_{n+p},t) \geq M(x_n,x_{n+1},\tfrac{t}{p}) * \ldots * M(x_{n+p-1},x_{n+p},\tfrac{t}{p}) \to 1*\ldots*1 = 1$$

as n →+∞. And by definition the sequence <x_n> is G-Cauchy. Since (X, M, *) is G-Complete. So this sequence <x_n> converges y ∈ X.

Now one can show that y = Ty i.e. y is fixed point of T as

$$\frac{1}{M(Ty,Tx_n,t)} - 1 \leq a \max\left\{\frac{1}{M(y,x_n,t)} - 1, \frac{1}{M(y,T\,y,t)} - 1, \frac{1}{M(x_n,x_{n+1},t)} - 1\right\}$$

Being the sequence <x_n> contractive as n → +∞, it follows that

$$\frac{1}{M(y,T\,y,t)} - 1 \leq a\left(\frac{1}{M(y,T\,y,t)} - 1\right)$$

for every t > 0. From this relation it follows that

$M(y,Ty,t) = 1$ for every t > 0 and so y = T(y).

To prove y = Sy, for every S ∈ F, by (1)

$$\frac{1}{M(Ty,Sy,t)} - 1 \leq a \max\left\{\frac{1}{M(y,y,t)} - 1, \frac{1}{M(y,T\,y,t)} - 1, \frac{1}{M(y,Sy,t)} - 1\right\}$$

from this, it follows that

$$\frac{1}{M(y,Sy,t)} - 1 \leq a\left(\frac{1}{M(y,S\,y,t)} - 1\right)$$

And so $M(y,Sy,t) = 1$ for every t > 0.

Consequently y = Sy.

Finally uniqueness follows as

Let z ∈ X be a common fixed point to all the mapping of the family F, by (1),

$$\frac{1}{M(Ty,Sz,t)} - 1 \leq a \max\left\{\frac{1}{M(y,z,t)} - 1, \frac{1}{M(y,T\,y,t)} - 1, \frac{1}{M(z,Sz,t)} - 1\right\}$$

$$\Rightarrow \frac{1}{M(y,z,t)} \leq a\left(\frac{1}{M(y,z,t)} - 1\right)$$

Consequently $M(y,z,t) = 1$, for every t > 0. so that y = z.

If in theorem 3, consider a fuzzy metric M_d induced by a metric d, it follows that the following relative result a family of mappings defined in a complete metric space.

Corollary 1; Let (X, d) be a complete metric space and let F be a family of mappings of X into itself. We suppose that there exists T ∈ F such that

d(Tx, Sy) ≤ a max $\{d(x,y), d(x,Tx), d(y,Sy)\}$

for every S ∈ F, for every x ∈ X. and y ∈ T(X) with 0 ≤ a < 1.

Then there exists a unique fixed point that is common to all the mappings of F.

Remark

If F = $\{T\}$. Then theorem (3), it follows that:-

Corollary 2. Let (X, M, ∗) be a G-Complete fuzzy metric space with M regular.

If the map T of X into itself is such that

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq a \max\left\{\frac{1}{M(x,y,t)} - 1, \frac{1}{M(x,T\,x,t)} - 1, \frac{1}{M(y,Ty,t)} - 1\right\}$$

for every x ∈ X and y ∈ T(X) with 0 ≤ a < 1. Then T has a unique fixed point.

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