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φ −Weak contractions in metric spaces

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Abstract
The existence of common fixed point for two mappings is proved in the metric space where one mapping is φ −weak contraction with respect to another mapping. The result generalizes a result of M. Abbas, M. Indad, and D. Gopal [20].

Keywords: φ −weak contraction, coincidence point, common fixed point.

1. Introduction

Banach contraction principle is one of the great results of modern analysis. The application of this great result is obviously seen in a number of branches of Mathematics. Generalization of this principal has been a heavily investigated branch of research. In [1], Boyd and Wong proved that the constant k can be replaced by the use of an upper semi continuous function. In [4], Suzuki has proved a generalization of the same principal. In [5], this principle is also extended to Menger space. In this paper, a result is generalized to metric space given by M. Abbas, M. Imdad, and D. Gopal on fuzzy metric space [10].
Khan *et al*. gave a fixed point theorem with the help of control function which is also called an altering distance function.

Definition 1.1 [6] A function $\psi: [0,\infty) \to [0,\infty)$is called an altering distance function if it satisfies the following properties:
(a) $\psi(0) = 0$,
(b) ψ is continuous and monotonically non-decreasing.

Theorem 1.2: [6] Let (X, d) be a complete metric space. Let ψ be an altering distance function, and let $f: X \to X$ be a self-mapping which satisfies the following inequality:

$$\psi(d(fx, fy)) \leq c\psi(d(x, y))$$

for all x, y $\in X$ and for some 0 < c < 1, then T has unique fixed point in X.
In fact khan *et al*. proved a more general theorem [6, theorem 2] of which the above result is corollary.

Definition 1.3 [9] A mapping $T: X \to X$, where (X, d) is a metric space, is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

where x, y $\in X$ and $\varphi: [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function such that $\varphi(t)$ if only if t = 0.

Definition 1.4 [10] Let (X, d) be a metric space and f: $X \to X$ be a map. The map $T: X \to X$ is called a φ −weak contraction with respect to f if there exists a function $\varphi: [0,\infty) \to [0,\infty)$ with $\varphi(r) > 0$ for r > 0 and $\varphi(0) = 0$ such that$d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy))$
for every x, y $\in X$. If f = I_x (an identity mapping on X), then T is called a φ −weak contraction.

Definition 1.5: [10] Let (X, d) be a metric space and f, T be two self-mappings on X. A point x in X is called a coincidence point (common fixed point) of f and T if fx = Tx (fx = Tx = x). Also the pair of mappings f, $T: X \to X$ are said to be weakly compatible if they commute on the set of coincidence points.

Theorem 1.6: [10] Let (X, M, *) be a fuzzy metric space and $T: X \to X$ be a ψ − weak contraction with respect to self mapping f on X. If the range of f contains the range of T and f(X) is a G−complete subspace of X, then f and T have coincidence point in X provided ψ is a continuous function.

Theorem 1.7: [9] If $T: X \to X$ is a weakly contractive mapping, where (X, d) be a complete metric space. Then T has a unique fixed point in X.

2. The Main Results

In this section, the two theorems are proved which follow as:
Theorem 2.1 Let (X, d) be a metric space and $T: X \to X$ be a φ –weak contraction with respect to self mapping f on X. If the range of f contains the range of T and f(X) is a complete subspace of X, then f and T have coincidence point in X provided that φ is a continuous mapping.

Proof Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $Tx_0 = fx_1$. This is possible since the range of f contains the range of T. Continuing in this way, indefinitely, for every x_n in X one can find a x_{n+1} such that $y_n = Tx_n = fx_{n+1}$. Without loss of generality, one may assume that $y_{n+1} \neq y_n$ for all $n \in N$, otherwise f and T have a common fixed point and there is nothing to prove.
In case $y_{n+1} \neq y_n$ we have,
$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1})$
$\leq d(fx_n, fx_{n+1}) - \varphi(d(fx_n, fx_{n+1}))$

This implies
$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n) - \varphi(d(y_{n-1}, y_n))$. (2.1)
Further this implies
$d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$. (2.2)
It follows that the sequence $\{d(y_n, y_{n+1})\}$ is a decreasing sequence and consequently there exists r $\geq$ 0 such that
$d(y_n, y_{n+1}) \to r$ as $n \to \infty$.
Letting $n \to \infty$ in (2.1)
$r \leq r - \varphi(r)$
which is a contradiction unless r = 0.
Hence
$d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. (2.3)

If possible, let $\{y_n\}$ be not a Cauchy sequence. Then there exists $\in > 0$ for which one can find sub sequence $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ with n(k) > m(k) > k such that
$d(y_{m(k),} y_{n(k)}) \geq \in$ (2.4)
Further, let n(k) is the smallest integer with n(k) > m(k) and satisfying (2.3).
Then
$d(y_{m(k),} y_{n(k)-1}) \geq \in$. (2.5)
Now
$\in \leq d(y_{m(k),} y_{n(k)}) \leq d(y_{m(k),} y_{n(k)-1}) + d(y_{n(k)-1,} y_{n(k)})$
$< \in + d(y_{n(k)-1,} y_{n(k)})$.
Letting $k \to \infty$
$\lim_{k\to\infty} d\big(y_{m(k),} y_{n(k)}\big) = \in$. (2.6)
Again,
$d(y_{n(k),} y_{m(k)})$
$\leq d(y_{n(k),} y_{n(k)-1}) + d(y_{n(k)-1,} y_{m(k)-1}) +$
$d(y_{m(k)-1,} y_{m(k)})$. (2.7)
and
$d(y_{n(k)-1,} y_{m(k)-1})$
$\leq d(y_{n(k)-1,} y_{n(k)}) + d(y_{n(k),} y_{m(k)}) + d(y_{m(k),} y_{m(k)-1})$.
(2.8)
Letting $k \to \infty$ in these two above inequality and using (2.3), (2.6), we obtain,
$d(y_{n(k)-1,} y_{m(k)-1}) \to \in$ as $k \to \infty$. (2.9)

Now from (2.4),
$\in \leq d(y_{m(k),} y_{n(k)}) = d(Tx_{m(k),} T\,x_{n(k)})$
$\leq d(fx_{m(k),} fx_{n(k)}) - \varphi(d\big(fx_{m(k),} fx_{n(k)}\big))$.
So,
$\in \leq d(y_{m(k)-1,} y_{n(k)-1}) - \varphi(d\big(y_{m(k)-1,} y_{n(k)-1}\big))$
Letting $k \to \infty$ implies that
$\in \leq \in - \varphi(r)$ which is a again contradiction as $\in > 0$.
This contradiction shows that $\{y_n\}$ is Cauchy sequence in f(X) and f(X) is complete. So, it converges to some point $y \in f(X)$ i.e. $y_n \to y \in f(X)$ as $n \to \infty$.
Consequently one obtain a point p in X such that fp = y. Next to show p is coincidence point of f and T, we have
$d(Tp, fx_{n+1}) = d(Tp, Tx_n) \leq d(fp, fx_n) - \varphi(d(fp, fx_n))$.
Letting, $n \to \infty$ we obtain,
$\lim_{n\to\infty} d(Tp, fx_{n+1}) \leq 0$.
This implies
$d(Tp, fp) \leq 0$.
This further implies
$0 \leq d(Tp, fp) \leq 0$.
Hence $d(Tp, fp) \leq 0$. This implies that Tp = fp. This completes the proof of the theorem.

Theorem 2.2 Let (X, d) be a metric space and T:X→X be a φ – weak contraction with respect to another self mapping f on X. If the range of f contains the range of T and f(X) is a complete metric subspace of X, then f and T have a common fixed point in X provided φ is continuous and the pair of mapping {f, T} is weakly compatible.

Proof By theorem 2.1, we obtain p in X such that Tp = fp = q (say). As the mappings are weak compatible. So fTp = Tfp. Obviously Tq = fq. Now we show that fq = q.
If not, then
$d(fq, q) = d(Tq, Tp) \leq d(fq, fp) - \varphi(d(fq, fp))$
$= d(fq, q) - \varphi(d(fq, q))$,
a contradiction. Therefore fq = q and Tq = fq = q
Now we prove the uniqueness of the theorem. For this let y be the another common fixed point of f and T. i.e. fy = Ty = y
$d(q, y) = d(Tq, Ty) \leq d(fq, fy) - \varphi(d(fq, fy))$
$= d(q, y) - \varphi(d(q, y))$,
a contradiction. This completes the proof of the theorem.

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