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# The oscillatory blood flow by viscosity shearing dependent model

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**Abstract**
Analysis of non-Newtonian flows in tubes is very important when studying the blood flow in different types of arteries. Usually the blood viscosity is defined by shearing-dependent models, which represents the viscosity as a non-linear function of the shear-rate. Present investigation was considered by oscillatory 2D model of the blood flow in a straight tube is discussed theoretically and numerically. The solution of the quasilinear parabolic equation for the velocity is constructed using appropriate analytical functions. Further the coresponding numerical solution is approximated by similar analytical functions.

**Keywords:** blood flow, viscosity, shearing depentent model

## Introduction
Investigations of hemodynamics in arteries is of primary importance for prediction and prevention of cardiovascular diseases. Apart the modern measurement technique. The mathetical modeling of the blood dynamics can help the betten understanding of such a phenomenon in a living body. The theoretical and numerical investigations can reveal important relations between the cardiovascular diseases and the real blood dynamics.
The blood is a non-Newtonian shear thinning fluid with a viscosity. which gradually decreases or increases with the shear rate increase of decrease reaching two different plateau values independent on the shear rate. The most common non-Newtonian rheological models used to describe the blood rheology are those of Carreau [1, 5], Carreau-Yasuda [1, 6, 9], Casson[1, 3, 8] and power law [1, 3, 4]. The models give a non-linear dependence of the shear stress on the shear rate. Since the shear rate changes significantly in the arteries. The viscosity cannot be assumed there as a constant. Thus, no analytical solutions of the blood flow in arteries could be found. The problem becomes more complicated if the pulsating character of the blood flow is taken into account.
For a pulsatile Newtonian flow in straight tubes the Womersley analytical solution is often used to approximate the blood flow in arteries [10]. The two-dimensional oscillatory Newtonian and non-Newtonian flows in straight and curved pipe geometries have been considered in [8]. The solutions have been obtained numerically by the lattice Boltzmann Equation. The Casson and Carreau-Yasuda models have been used for the blood viscosity description. The authors find that the velocity and shear stress differ insignificantly from their respective due to newtonian flow, but this difference increases for smaller Womersley numbers.

## Materials and Methods
The equations of motion and continuity eqution in vector form for a non-Newtonian and Newtonian fluid are:

$$\rho(\frac{\partial v}{\partial t} + \mathrm{v} \cdot \nabla \mathrm{v}) = -\nabla p + \nabla \cdot \mathrm{T}, \qquad \text{....... (1)}$$

$$\nabla \cdot \mathrm{v} = 0 \qquad \text{...... (2)}$$

Where, $v = (v_x, v_y)$ is the velocity vector, $p$ is the pressure and T = $f(\dot{S})$ is the viscous stress tensor, where, $\dot{S}$ is the shear rate tensor. Since the tube is assumed as infinitely long, the velocity will depend only on the width coordinate $y$ and time $t$.
Moreover, as the pressure changes only in the axial direction x and in time t, from equation (1) and (2) it follows that $v_y = 0$ and only non-zero terms of T is $T_{xy} = T_{yx}$ $=\mu_{app}(\dot{\gamma})\dot{\gamma}$. where $\dot{\gamma} = \frac{\partial v_x}{\partial y}$ is the shear rate and $\mu_{app}$ is the so called apparent fluid viscosity. Then The equation (1) and (2) transform into one single equation for $v_x$:

$$\rho\frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu_{app}\frac{\partial v_x}{\partial y}\right) \qquad \text{...... (3)}$$

The boundary conditions are the no-slip once along the tube walls, i.e., at $v_x = 0$ at $y = 0$ and $y = H.$ The apparent viscosity $\mu_{app}$ for the Carreau viscosity model, further denoted as $\mu_c$, is given by the relation [2]:

$$\mu_c = \mu_\infty + (\mu_0 - \mu_\infty)[1 + \lambda^2\gamma^2]^{(n_c-1)/2}, \qquad \text{....(4)}$$

Where, $\mu_0$, $\mu_\infty$ are the upper and lower limits of the viscosity corresponding to the low and high shear rates and the coefficients λ and $n_c$ are empirically determined. For human blood the values of these parameters are the following [2]:

$\mu_0 = 0.056\ Pa\ s,\ \mu_\infty = 0.00345\ Pa\ s,\ \lambda = 3.313s$

and $n_c$ = 0.3568. The dimentionless variant of eq.(3), after using H as a characteristic length (y= HY) and 1/n as acharacteristic time (t = T/n), becomes:

$$\alpha^2 \frac{\partial u}{\partial t} - \frac{\partial}{\partial y}\left(\bar{\mu}_{app} \frac{\partial}{\partial y}\right) - B\cos(T) = 0, \quad \text{..... (5)}$$

Where, $u = v_x$, $\alpha = H\sqrt{\frac{\rho^n}{\mu_\infty}}$ is the Womersley number [10]. $B = \frac{AH^2}{\mu_\infty}$, $\bar{\mu}_{app} = \mu_{app}/\mu_\infty$ and $\mu_\infty$ is the viscosity of the blood asumed as a Newtonian fluid. Thus, for the Newtonian viscosity model $\bar{\mu}_{app} = 1$. The dimensionless boundary conditions are shown in equation (6):

$u(0,T) = u(1,T) = 0$ .....(6)

The analytical solution from equation (5) and (6) (a linear boundary value problem for the Newtonian viscosity model velocity at $\bar{\mu}_{app}$=1), is well known [8] and further denoted as $v(Y,T)$ in equation (7):

$$v(Y,T) = \frac{B}{\alpha^2}[E(Y)\sin T + D(Y)\cos T] \quad \text{..... (7)}$$

where:

$$E(Y) = 1 + \frac{1}{1-\cos^2\beta - \cosh^2\beta}[S_1(Y)S_2 + C_1(Y)C_2] \quad \text{.. (8)}$$

$$D(Y) = 1 + \frac{1}{1-\cos^2\beta - \cosh^2\beta}[S_1(Y)C_2 + C_1(Y)S_2] \quad \text{..... (9)}$$

With $S_1(Y) = \sin 2\beta\left(Y - \frac{1}{2}\right)\sinh 2\beta\left(Y - \frac{1}{2}\right), C_1(Y) = \cos 2\beta\left(Y - \frac{1}{2}\right)\cosh 2\beta\left(Y - \frac{1}{2}\right), S_2 = \sin\beta\sinh\beta, C_2 = \cos\beta\cosh\beta, \beta = \frac{\sqrt{2\alpha}}{4}$.

**Theoretical analysis of the non-Newtonian problem:**
In this section we consider the non-Newtonian boundary value problem from equation (5) and (6) with $\bar{\mu}_{app} = \bar{\mu}_c = 1 + C[1 + \lambda_H^2 u_Y^2]^{(n_c-1)/2}$, $\lambda_H = \frac{\lambda}{H}$ and $C = \frac{\mu_0}{\mu_\infty} - 1$.

Then the problem can be rewritten in the following manner as shown in equation (10-13):

$$P(u) = \alpha^2 u_T - \left[1 + C(1 + n_c\lambda_H^2 u_Y^2)(1 + \lambda_H^2 u_Y^2)^{\frac{n_c-3}{2}}\right]u_{YY} - B\cos(T) = 0 \quad \text{...(10)}$$

in $Q = [0.2m\pi] \times [0.1]$,

$$u(0,T) = u(1,T) = 0 \quad \text{....(11)}$$

For $T \in [0.2m\pi]$,

$$u(Y,0) = \varphi(Y) \quad \text{....(12)}$$

For $Y \in [0,1]$. Here m> 0 is an integer number, $\alpha, C, \lambda, n_c$ are positive constants, 0<$n_c < 1$ and

$$\psi(Y) \in C^3([0,1]), \psi(0) = \psi(1) = 0. \quad \text{......(13)}$$

Since $1 \le 1 + C(1 + \lambda_H^2 n_c u_Y^2)(1 + \lambda_H^2 u_Y^2)^{\frac{n_c-3}{2}} \le 1 + C$, the equation (10) is a quasilinear parabolic equation, According to [11], the boundary value problem from equation (10), (12) has an unique classical solution $u(Y,T) \in C^2(Q)$. Moreover, the following comparison principle for equation (10) and (11) holds for its proof [11].
Theorem 1 (Comaparison principle).

Suppose,

$u_1(Y,T)$, is a classical solution of (10),(11) and $u_2(Y,T) \in C^2(Q)$

Satisfy the inequalities:
$u_1(0,T) \le u(0,T) \le u_2(0,T)$ for $T \in [0.2m\pi]$,
$u_1(1,T) \le u(1,T) \le u_2(1,T)$ for $T \in [0.2m\pi]$,
$u_1(Y,0) \le u(Y,0) \le u_2(Y,0)$ for $Y \in [0.1]$,
and $P(u_1) \le P(u) \le P(u_2)$ in $\bar{Q}$.

Then $u_1(Y,T) \le u(Y,T) \le u_2(Y,T)$ for every $(Y,T) \in \bar{Q}$.

By means of the therom 1 we get the following a prioriestimates for the solution $u(Y,T)$ of (10), (11), which are presented in the following therom:

and $u_1(Y,0) = -\frac{B}{\alpha^2} \le \psi(Y) = u(Y,0)$ for $Y \in [0,1]$.

**Results and Discussion**

The blood density $\rho$ is assumed to be $\frac{1000kg}{m^3}$, the amplitude of the oscillatory pressure gradient- $A = 6000\frac{Pa}{m}$,the angular frequency – $n = 2\pi f$, where $f = 1.2$ is the normal pulse frequency and the tube width (diameter) $H = 0.009m$. This tube size corresponds to a size a human carotid artery. For these values of the physical parameters the Womersley number is $\alpha = 13.305$.
With these data from eq. (5) together with the conditions in equation (6) is solved numerically by the Crank-Nicholson method in combination with the Picard successive iteration method. The time and space steps are $O(10^{-3})$, ensuring an relative error of $O(10^{-3})$, for the velocity. The equation (5) is solved iteratively starting from the analytical Newtonian solution $v(Y,T)$ given by equation (7). According to the analysis in the previous chapter. if the initial condition for $u(Y,T)$ is bounded and is not far from the newtonian solution $u(Y,T)$. The numerical solution, futher denoted by $u(Y,T)$ has been approximated by an analytical function similar to the Newtonian velocity $v(Y,T)$ as shown in equation (14):

$$u_n(Y,T) = \frac{B}{\alpha^2}[E_n(Y)\sin T + D_n(Y)\cos T] \quad \text{...... (14)}$$

where:

$$E_n(Y) = \alpha_0 + \alpha_1 d_1(Y)d_2(Y)_+ \alpha_2 d_3(Y)d_4(Y)$$

$$D_n(Y) = \alpha_3 d_1(Y)d_2(Y)_+ \alpha_1 d_3(Y)d_1(Y)$$

where

$$d_1(Y) = \cos[c\left(Y - \frac{1}{2}\right), d_2(Y) = \cosh\left[c\left(Y - \frac{1}{2}\right)\right], d_3(Y) = sin\left[c\left(Y - \frac{1}{2}\right)\right], d_4(Y) = sinh\left[c\left(Y - \frac{1}{2}\right)\right]$$

Are functions of Y and $\alpha_0, \alpha_1, \alpha_2, b_1, b_2$
And c are constants fitted with the numerical solution.
The velocity correspondent to the maximum 2D flow rate $Q_{max}\, 0.0064m^2$/s, reached at $t = 0.192s\ (T = 0.23 \times 2\pi)$. The velocity correspondent to the minimum (in the positive or systole time cycle at $T = 0.48 \times 2\pi)$ flow rate is presented. The Carreau velocity profiles are slightly flattened with respect to the Newtonian profiles. Similar velocity profiles have been obtained in[8] for a similar problem, but using shearing dependent model for blood viscosity. The most important characteristics of the blood flow in arteries is the shear stress distribution in the tube and mainly on the tube walls. The absolute value of the WSS (wall shear stress), $\mu\partial v\partial y$ $y$=0 in function of time for one period is almost the same as the Newtonian ones. The peak WSS is 4.05$Pa$ for the Carreau model and 4.06$Pa$ for the Newtonian model. These values of the peak WSS are in the limits of the experimentally registered WSS [12] in a human carotid, which are 2.5-4.3$Pa$. If the tube is considered as axisymmetrical, a similar analysis gives the peak WSS equal to 3.97$Pa$ for the shearing dependent model and to 3.85$Pa$ for the Newtonian model.

## Conclusion

The non-Newtonian blood flow in a 2D variant of a straight tube has been studied theoretically and numerically, using the Carreau viscosity model. The pressure gradient along the tube is assumed pulsatile with a constant amplitude. The equation of motion is from the class of quasilinear parabolic equations that have been intensively studied for existence and uniqueness by many authors [11]. In the present work it is shown that the solutions of the equation of motion are bound from below and from above by subsolutions and supersolutions (barrier functions). Further it is shown that the difference between the solutions of the Newtonian and Carreau viscosity model flow is also bound. The Newtonian flow velocity solution is analytical, while the non-Newtonian flow equation has no analytical solution. There exist different numerical methods to find the non-Newtonian flow solution [8]. Here the problem is solved numerically in finite-differences by the Crank-Nicholson method with successive iterations for its linearization. Then the obtained solution is approximated by analytical functions in a similar manner as the Newtonian solution. It is shown that the velocity and the WSS obtained by the Newtonian and shearing dependent model are close.

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