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A search on the integer solutions of pell-like equation $ax^2-(a-1)y^2=a,\ a>1$

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Abstract
This paper deals with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous binary quadratic equation represented by the Pell-like equation $ax^2-(a-1)y^2=a,\ a>1$. Different sets of integer solutions are presented. For illustration, the integer solutions to the above equation when a=11 are presented. The construction of second order Ramanujan Numbers is illustrated. Employing the solutions, a few relations among special polygonal numbers are obtained.

Keywords: non homogeneous binary quadratic, pell-like equation, hyperbola, integral solutions, special numbers

Introduction
The binary quadratic Diophantine equations of the form $ax^2-by^2=N,(a,b,N\neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a,b and N. In this context, one may refer [1, 17]. This paper deals with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous binary quadratic equation represented by the Pell-like equation $ax^2-(a-1)y^2=a,a>1$. Different sets of integer solutions are presented. For illustration, the integer solutions to the above equation when a=11 are presented. In this example, the construction of second order Ramanujan Numbers with base numbers as real integers and Gaussian integers is illustrated and employing the solutions, a few relations among special polygonal numbers are obtained. A special Pythagorean triangle is also determined.

Method of Analysis
Let a (>1) be any positive integer. The Pell-like equation under consideration is

$$ax^2-(a-1)y^2=a,a>1 \tag{1}$$

The process of obtaining different choices of integer solutions to (1) is illustrated below:

Choice (1)
Taking

$$x=2k+1,y=2s \tag{2}$$

in (1), it is written as

$$a(k^2+k)=(a-1)s^2 \tag{3}$$

which is satisfied by

$$k=a-1,s=a \tag{4}$$

And

$$k=-a,s=a \tag{5}$$

In view of (2), the integer solutions to (1) are given by

$$x = 2a - 1, y = 2a \quad (6)$$

and

$$x = -2a + 1, y = 2a \quad (7)$$

It is worth mentioning that, in (6), the values of x and y are consecutive integers, where as in (7), the sum x+y=1

Choice (2)
The substitution

$$y = aT \quad (8)$$

In (1) leads to

$$x^2 = a(a-1)T^2 + 1 \quad (9)$$

which is the well-known pellian equation whose general solution is given by

$$T_n = \frac{1}{2\sqrt{a(a-1)}} g_n$$

$$x_n = \frac{1}{2} f_n \quad (10)$$

where

$$f_n = (2a - 1 + 2\sqrt{a(a-1)})^{n+1} + (2a - 1 - 2\sqrt{a(a-1)})^{n+1}$$

$$g_n = (2a - 1 + 2\sqrt{a(a-1)})^{n+1} - (2a - 1 - 2\sqrt{a(a-1)})^{n+1}$$

In view of (8), one obtains

$$y_n = \frac{a}{2\sqrt{a(a-1)}} g_n \quad (11)$$

Thus, (10) and (11) represent the integer solutions to (1)

Choice (3)

Introducting the linear transformations

$$x = X + (a-1)T, y = X + aT \quad (12)$$

In (1), it becomes

$$X^2 = a(a-1)T^2 + a \quad (13)$$

whose least positive integer solution is

$$T_0 = 1, X_0 = a$$

To obtain the other solutions of (13), consider the Pell equation

$$X^2 = a(a-1)T^2 + 1$$

whose general solution is given by

$$T_n = \frac{1}{2\sqrt{a(a-1)}} g_n$$

$$X_n = \frac{1}{2} f_n$$

where

$$f_n = (2a-1+2\sqrt{a(a-1)})^{n+1} + (2a-1-2\sqrt{a(a-1)})^{n+1}$$

$$g_n = (2a-1+2\sqrt{a(a-1)})^{n+1} - (2a-1-2\sqrt{a(a-1)})^{n+1}$$

Applying Brahmagupta lemma between (X_0, T_0) and $(X_n\ , T_n)$, the other integer solutions of (13) are given by

$$T_{n+1_n} = X_n + aT_n$$

$$X_{n+1} = aX_n + a(a-1)T_n$$

In view of (12), the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x_{n+1} &= (2a-1)X_n + 2a(a-1)T_n \\ y_{n+1} &= 2aX_n + a(2a-1)T_n \end{aligned}\right\}, \ n = -1,0,1.......$$

The recurrence relations satisfied by x and y are given by

$$\left.\begin{aligned} x_{n+3} - (4a-2)x_{n+2} + x_{n+1} &= 0 \\ y_{n+3} - (4a-2)y_{n+2} + y_{n+1} &= 0 \end{aligned}\right\}, n = -1,0,1,.....$$

To analyze the nature of solutions, one has to go for particular values of a.
For illustration, the choice a=11 in (1) leads to

$$11x^2 - 10y^2 = 11 \tag{14}$$

whose general solution is given by

$$x_{n+1} = \frac{21}{2} f_n + \sqrt{110} g_n$$

$$y_{n+1} = 11f_n + \frac{1}{2}\frac{231}{\sqrt{110}} g_n$$

where

$$f_n = (21+2\sqrt{110})^{n+1} + (21-2\sqrt{110})^{n+1}$$

$$g_n = (21+2\sqrt{110})^{n+1} - (21-2\sqrt{110})^{n+1}$$

The recurrence relations satisfied by x and y are given by

$$\left.\begin{aligned} x_{n+3} - 42x_{n+2} + x_{n+1} &= 0 \\ y_{n+3} - 42y_{n+2} + y_{n+1} &= 0 \end{aligned}\right\}, N = -1,0,1.......$$

Some numerical examples of x and y satisfying (14) are given in the Table: 1 below:

Table 1: Numerical Examples

n	**x_{n+1}**	**y_{n+1}**
-1	21	22
0	881	924

1	36981	38786
2	1552321	1628088
3	65160501	68340910

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} is odd& y_{n+1} is even.
2. $x_{2n} \equiv 0(\mod 3), n = 0,1,2.......$
3. $y_{2n-1} \equiv 0(\mod 4), n = 1,2,3.......$

Observations

a. From the suitable values of x_{n+1}, y_{n+1} , one may generate second order Ramanujan numbers with base numbers as real integers as well as gaussian integers.

Illustration 1

Consider $x_0 = 21 = 1*21 = 3*7$ (15)

$$= 11^2 - 10^2 = 5^2 - 2^2$$

$$\Rightarrow 11^2 + 2^2 = 10^2 + 5^2 = 125$$

Also, from (15), we have

$$(21+1)^2 + (7-3)^2 = (21-1)^2 + (7+3)^2$$

$$\Rightarrow 22^2 + 4^2 = 20^2 + 10^2 = 500$$

Thus, 125 and 500 are second order Ramanujan numbers with base numbers as real integers.
Further, from (15), one may write

$$(21+i)^2 + (7-3i)^2 = (21-i)^2 + (7+3i)^2 = 480$$

Here, 480 is second order Ramanujan numbers with base numbers as Gaussian integers.

Illustration 2

$$y_0 = 22 = 1*22 = 2*11$$

$$(22+i)^2 + (11-2i)^2 = (22-i)^2 + (11+2i)^2 = 600$$

$\therefore$ 600 is the second order Ramanujan number with base numbers as gaussian integers.
It is worth to note that it is not possible to get second order Ramanujan numbers with base numbers as real integers since 22 cannot be expressed as the difference of two integer squares in two different ways.

b. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1} = \frac{x_{n+1}-1}{2}, b_{n+1} = \frac{y_{n+1}}{2}$

Then, the following relations are observed:

a. $1320 t_{3,a_{n+1}}$ is a Nasty Number.

b. $88 t_{3,a_{n+1}} - t_{82,b_{n+1}} \equiv 0(\mod 13)$

c. $88 t_{3,a_{n+1}} - t_{52,,b_{n+1}} - t_{32,b_{n+1}} \equiv 0(\mod 19)$

c. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where

$$a_{n+1} = \frac{x_{n+1}+1}{2}, b_{n+1} = \frac{y_{n+1}}{2}$$

1. $110(S_{a_{n+1}} - 1)$ is a Nasty Number.
2. $11 S_{a_{n+1}} = 11 + 60 b_{n+1}^2$

d. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where

$$a_{n+1} = \frac{x_{n+1} - 3}{2}, b_{n+1} = \frac{y_{n+1}}{2}$$

1. $-2CD_n + t_{22,b_{n+1}} + 9b_{n+1} + a_{n+1} + 124$ is a perfect square
2. $CH_n + 11a_{n+1} + 10 = 5b_{n+1}^2$

e. Consider $p = x_{n+1} + y_{n+1}$, $q = y_{n+1}$ Note that $p > q > 0$ .Treat p, q as the generators of the Pythagorean triangle $T(X, Y, Z)$ where

$$X = 2pq, Y = p^2 - q^2, z = p^2 + q^2$$

Then the following results are obtained:

a. 11X-5Y-6Z+11=0

b. $\frac{2A}{P} = x_{n+1} y_{n+1}$, where A=area and P=Perimeter.

c. $3\left(X - \frac{4A}{P}\right)$ is a nasty number.

d. $X - \frac{4A}{P} + Y$ is written as the sum of two squares.

Note 1: One may also consider the linear transformations

$$x = X - 10T \text{ , } y = X - 11T$$

which lead to a different set of integer solutions to (14).

Note 2: The substitution

$$y = 11T$$

in (14) leads to

$$x^2 = 110T^2 + 1$$

which is the well-known pellian equation whose general solution is given by

$$T_n = \frac{1}{2\sqrt{110}} g_n \text{ , } x_n = \frac{1}{2} f_n$$

In view of (6), one obtains

$$y_n = \frac{11}{2\sqrt{110}} g_n$$

Thus, the above values of x_n, y_n satisfy (14) and these are different from the solutions presented above.

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