International Journal of Advanced Scientific Research
ISSN: 2456-0421, Impact Factor: RJIF 5.32
www.newresearchjournal.com/scientific
Volume 1; Issue 2; May 2016; Page No. 62-64

On quadratic Diophantine equation with five unknowns $4w^2 - x^2 - y^2 + z^2 = t^2$

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Abstract
This paper aims at presenting general formulas for generating integer solutions of the quadratic equation in title based on its given integer solution.

Keywords: Ternary quadratic, integer solutions, figurate numbers, homogenous quadratic, polygonal Numbers and pyramidal numbers

Introduction

The Diophantine equations offer an unlimited field for due to their variety [1, 3]. In particular, one may refer [4-16] for quadratic equations with three unknowns. This communication concerns with yet another interesting homogenous quadratic equation with three unknowns $4w^2 - x^2 - y^2 + z^2 = t^2$ for determining it's infinitely many non –zero integral points. Also, a few interesting relations among the solutions are presented.

Method of Analysis

To start with, it is noted that the quadratic Diophantine with five unknowns represented by

$$4w^2 - x^2 - y^2 + z^2 = t^2 \tag{1}$$

Is satisfied by the values of x, y, z, w and t presented in the table below:

Table 1: Solutions

x	y	z	w	t
$(4uv)$	α	α	(u^2+v^2)	$2(u^2-v^2), u>v>0$
$2(u^2-v^2)$	α	α	(u^2+v^2)	$(4uv)$
$(8uv)$	α	α	$2(u^2+v^2)$	$4(u^2-v^2)$
$4(u^2-v^2)$	α	α	$2(u^2+v^2)$	$(8uv)$
$2(4uv-2u+2v+1)$	α	α	$(2u^2+2u+2v^2+2v+1)$	$4(u-v)(u+v+1)$
$4(u-v)(u+v+1)$	α	α	$(2u^2+2u+2v^2+2v+1)$	$2(4uv-2u+2v+1)$
$(2uv)$	(u^2-v^2)	(u^2+v^2)	α	$\pm 2\alpha$

From the above table it is observed that (1) is satisfied by infinitely many non-zero integer solutions. Now, a natural question may arise namely, given an integer solution of (1), whether it is possible to obtain a general formula for obtaining sequence of integer solutions based on the given solutions. The answer to this question is in the affirmative. In what follows, we exhibit two different general formulas for generating sequences of integer solutions for the given equation based on its known solutions.

Generation of integer solution

Let $(x_0, y_0, z_0, w_0, t_0)$ be any given solution of (1)

Formula 1

Let $x_1 = h - x_0, y_1 = h - y_0, t_1 = h - t_0, z_1 = z_0 + h, w_1 = w_0$ (2)

Be the second solution of (1), substituting (2) in (1) and simplifying, we get

$$h = x_0 + y_0 + z_0 + t_0$$

The second solution of (1) expressed in the matrix form is

$$(x_1, y_1, z_1, t_1)^T = \begin{bmatrix} a_{11}^{(1)}, & a_{12}^{(1)}, & a_{13}^{(1)}, & a_{14}^{(1)} \\ a_{21}^{(1)}, & a_{22}^{(1)}, & a_{23}^{(1)}, & a_{24}^{(1)} \\ a_{31}^{(1)}, & a_{32}^{(1)}, & a_{33}^{(1)}, & a_{34}^{(1)} \\ a_{41}^{(1)}, & a_{42}^{(1)}, & a_{43}^{(1)}, & a_{44}^{(1)} \end{bmatrix}, w_1 = w_0$$

Where

$$a_{11}^{(1)} = a_{22}^{(1)} = a_{33}^{(1)} = 0,$$
$$a_{12}^{(1)} = a_{13}^{(1)} = a_{14}^{(1)} = a_{21}^{(1)} = a_{23}^{(1)} = a_{24}^{(1)} = a_{31}^{(1)} = a_{32}^{(1)} = a_{34}^{(1)} = a_{41}^{(1)} = a_{42}^{(1)} = a_{43}^{(1)} = 1,$$
$$a_{44}^{(1)} = 2$$

And T is transpose
Repeating the above process, the general formula for obtaining other integer solutions of (1) is given by

$$(x_n, y_n, z_n, t_n)^T = \begin{bmatrix} a_{11}^{(n)}, & a_{12}^{(n)}, & a_{13}^{(n)}, & a_{14}^{(n)} \\ a_{21}^{(n)}, & a_{22}^{(n)}, & a_{23}^{(n)}, & a_{24}^{(n)} \\ a_{31}^{(n)}, & a_{32}^{(n)}, & a_{33}^{(n)}, & a_{34}^{(n)} \\ a_{41}^{(n)}, & a_{42}^{(n)}, & a_{43}^{(n)}, & a_{44}^{(n)} \end{bmatrix}, w_n = w_0$$

Where

$$a_{11}^{(n)} = a_{22}^{(n)} = a_{33}^{(n)} = a_{12}^{(n-1)} + a_{13}^{(n-1)} + a_{14}^{(n-1)},$$
$$a_{12}^{(n)} = a_{13}^{(n)} = a_{21}^{(n)} = a_{23}^{(n)} = a_{31}^{(n)} = a_{32}^{(n)} = a_{11}^{(n)} - (-1)^n,$$
$$a_{14}^{(n)} = a_{24}^{(n)} = a_{34}^{(n)} = a_{41}^{(n)} = a_{42}^{(n)} = a_{43}^{(n)} = a_{11}^{(n-1)} + a_{12}^{(n-1)} + a_{13}^{(n-1)} + 2a_{14}^{(n-1)}$$
$$a_{44}^{(n)} = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)}, n \geq 2$$

Formula 2

Let $x_1 = x_0 + h, y_1 = y_0 + h, t_1 = t_0 + h, z_1 = h - z_0, w_1 = h - w_0$ (3)

Be the second solution of (1), substituting (3) in (1) and simplifying, we get

$$h = x_0 + y_0 + z_0 + t_0 + 4w_0$$

The second solution of (1) expressed in the matrix form is

$$(x_1, y_1, z_1, t_1, w_1)^T = \begin{bmatrix} a_{11}^{(1)}, & a_{12}^{(1)}, & a_{13}^{(1)}, & a_{14}^{(1)}, & a_{15}^{(1)} \\ a_{21}^{(1)}, & a_{22}^{(1)}, & a_{23}^{(1)}, & a_{24}^{(1)}, & a_{25}^{(1)} \\ a_{31}^{(1)}, & a_{32}^{(1)}, & a_{33}^{(1)}, & a_{34}^{(1)}, & a_{35}^{(1)} \\ a_{41}^{(1)}, & a_{42}^{(1)}, & a_{43}^{(1)}, & a_{44}^{(1)}, & a_{45}^{(1)} \\ a_{51}^{(1)}, & a_{52}^{(1)}, & a_{53}^{(1)}, & a_{54}^{(1)}, & a_{55}^{(1)} \end{bmatrix}$$

Where

$$a_{11}^{(1)} = a_{22}^{(1)} = a_{44}^{(1)} = 0,$$
$$a_{12}^{(1)} = a_{13}^{(1)} = a_{14}^{(1)} = a_{21}^{(1)} = a_{23}^{(1)} = a_{24}^{(1)} = 1$$
$$a_{31}^{(1)} = a_{32}^{(1)} = a_{34}^{(1)} = a_{41}^{(1)} = a_{42}^{(1)} = a_{43}^{(1)} = a_{51}^{(1)} = a_{52}^{(1)} = a_{53}^{(1)} = a_{54}^{(1)} = 1$$
$$a_{15}^{(1)} = a_{25}^{(1)} = a_{35}^{(1)} = a_{45}^{(1)} = 4, a_{55}^{(1)} = 3$$

And T is transpose
Repeating the above process, the general formula for obtaining other integer solutions of (1)
Is given by

$$(x_n, y_n, z_n, t_n, w_n)^T = \begin{bmatrix} a_{11}^{(n)}, & a_{12}^{(n)}, & a_{13}^{(n)}, & a_{14}^{(n)}, & a_{15}^{(n)} \\ a_{21}^{(n)}, & a_{22}^{(n)}, & a_{23}^{(n)}, & a_{24}^{(n)}, & a_{25}^{(n)} \\ a_{31}^{(n)}, & a_{32}^{(n)}, & a_{33}^{(n)}, & a_{34}^{(n)}, & a_{35}^{(n)} \\ a_{41}^{(n)}, & a_{42}^{(n)}, & a_{43}^{(n)}, & a_{44}^{(n)}, & a_{45}^{(n)} \\ a_{51}^{(n)}, & a_{52}^{(n)}, & a_{53}^{(n)}, & a_{54}^{(n)}, & a_{55}^{(n)} \end{bmatrix}$$

Where

$$\begin{aligned}
&a_{11}^{(n)} = a_{22}^{(n)} = a_{44}^{(n)} = 2a_{11}^{(n-1)} + a_{12}^{(n-1)} + a_{13}^{(n-1)} + a_{14}^{(n-1)} + a_{15}^{(n-1)}, \\
&a_{12}^{(n)} = a_{14}^{(n)} = a_{21}^{(n)} = a_{24}^{(n)} = a_{41}^{(n)} = a_{42}^{(n)} = a_{11}^{(n)} - 1, \\
&a_{15}^{(n)} = 4\left(a_{11}^{(n-1)} + a_{12}^{(n-1)} + a_{13}^{(n-1)} + a_{14}^{(n-1)}\right) + 3a_{15}^{(n-1)} = a_{25}^{(n)} = a_{45}^{(n)}, \\
&a_{13}^{(n)} = \frac{a_{15}^{(n)}}{4} = a_{23}^{(n)} + a_{31}^{(n)} + a_{32}^{(n)} + a_{34}^{(n)} + a_{43}^{(n)} + a_{51}^{(n)} + a_{52}^{(n)} + a_{54}^{(n)} \\
&a_{33}^{(n)} = a_{31}^{(n-1)} + a_{32}^{(n-1)} + a_{34}^{(n-1)} + a_{35}^{(n-1)}, \\
&a_{35}^{(n)} = 4\left(a_{31}^{(n-1)} + a_{32}^{(n-1)} + a_{33}^{(n-1)} + a_{34}^{(n-1)}\right) + 3a_{35}^{(n-1)}, \\
&a_{53}^{(n)} = a_{33}^{(n)} - (-1)^n, \\
&a_{55}^{(n)} = a_{35}^{(n)} + (-1)^n
\end{aligned}$$

Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $4w^2 - x^2 - y^2 + z^2 = t^2$. As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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