International Journal of Advanced Scientific Research
ISSN: 2456-0421, Impact Factor: RJIF 5.32
www.newresearchjournal.com/scientific
Volume 1; Issue 2; May 2016; Page No. 59-61

On ternary quadratic Diophantine equation $5(x^2 + y^2) - 6xy = 20z^2$

[1] Meena K, [2] Vidhyalakshmi S, [3] Bhuvaneswari E, [4] Presenna R

[1] Former VC, Bharathidasan University, Trichy, Tamil Nadu, India
[2] Professor, Department of Mathematics, SIGC, Trichy, Tamil Nadu, India
[3] MPhil Scholar, Department of Mathematics, SIGC, Trichy, Tamil Nadu, India
[4] Research Scholar, Dept. of mathematics, SIGC, Trichy, Tamil Nadu, India

Abstract

The ternary homogeneous quadratic equation given by $5(x^2 + y^2) - 6xy = 20z^2$ representing a cone is analysed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented.

Keywords: Ternary quadratic, integer solutions, figurate numbers, homogenous quadratic, polygonal Numbers and pyramidal numbers

Introduction

The Diophantine equations offer an unlimited field for due to their variety [1, 3]. In particular, one may refer [4, 16] for quadratic equations with three unknowns. This communication concerns with yet another interesting homogenous quadratic equation with three unknowns $5(x^2 + y^2) - 6xy = 20z^2$ for determining it's infinitely many non – zero integral points. Also, a few interesting relations among the solutions are presented.

Method of Analysis

The Diophantine equation representing the ternary quadratic equation to be solved for its non-zero distinct integral solutions is

$$5(x^2 + y^2) - 6xy = 20z^2 \tag{1}$$

The substitution of linear transformations

$$\left.\begin{aligned} x &= u + v \\ y &= u - v \end{aligned}\right\} , \quad u \neq v \neq 0 \tag{2}$$

in (1) leads to,

$$u^2 + (2v)^2 = 5z^2 \tag{3}$$

$$\Rightarrow p^2 + q^2 = 5z^2 \tag{4}$$

Where

$$p = u, q = 2v \tag{5}$$

Assume that

$$z = a^2 + b^2 , \quad a, b \neq 0 \tag{6}$$

Pattern 1

Write 5 as

$$5 = (2 + i)(2 - i) \tag{7}$$

Substituting (7) and (6) in (4) and using the method of factorization, we get

$$(p + iq)(p - iq) = (2 + i)(2 - i)(a + ib)^2 (a - ib)^2$$

equating the positive and negative factors, the resulting equations are

$$p + iq = (2 + i)(a + ib)^2 \tag{8}$$

$$p - iq = (2 - i)(a - ib)^2 \tag{9}$$

Equating the real and imaginary parts in (8)

$$\begin{aligned} p &= 2a^2 - 2b^2 - 2ab \\ q &= a^2 - b^2 + 4ab \end{aligned} \tag{*}$$

In view of (5),

$$\begin{aligned} u &= 2a^2 - 2b^2 - 2ab \\ v &= \frac{a^2}{2} - \frac{b^2}{2} + 2ab \end{aligned} \tag{10}$$

Substituting (10) in (2), we get

$$\begin{aligned} x &= \frac{5a^2}{2} - \frac{5b^2}{2} \\ y &= \frac{3a^2}{2} - \frac{3b^2}{2} - 4ab \end{aligned} \tag{11}$$

As our interest is on finding integer solutions, we choose a and b suitably so that the values of x and y are in integers. Replacing a by 2A and b by 2B in (11) & (6), the integer solutions to (1) are

$$x(A, B) = x = 10A^2 - 10B^2$$

$$y(A, B) = y = 6A^2 - 6B^2 - 16AB$$

$$z(A, B) = z = 4A^2 + 4B^2$$

Properties

$1.6x(n^2,n+1)-10y(n^2,n+1)=320p_n^5$

$2.6x(n,n+1)-10y(n,n+1)=320t_{3,n}$

$3.6x(n,n^2)-10y(n,n^2)=160cp_{6.n}$

$4.6x(n(n+1),n+2)-10y(n(n+1),n+2)=960p_n^3$

Pattern 2

Instead of (7), 5can be written as

$$5=(1+2i)(1-2i) \quad (12)$$

Proceeding as in method 1, the non-zero distinct integral solutions to (1) are given by

$$x(a,b)=x=2a^2-2b^2-3ab$$
$$y(a,b)=y=-5ab$$
$$z(a,b)=z=a^2+b^2$$

Properties

1) $15[5x(a,b)-3y(a,b)+10t_{4,b}]$ is a nasty number
2) $2z(n,n+1)-x(n,n+1)-6t_{3,n}$ is a perfect square
3) $6[2z(n,n+1)-x(n,n+1)-6t_{3,n}]$ is a nasty number
4) $[x(a,a)-y(a,a)]$ is a perfect square
5) $3[x(a,a)-y(a,a)]$ is a nasty number

In addition to (7) and (12), 5 may also be represented as

Set 1: $5=(-2+i)(-2-i)$

Set 2: $5=(-1+2i)(-1-2i)$

Proceeding as in method 1, two more choices of integer solutions to (1) are presented below.

Solutions of Set 1:

$$x(A,B)=y=-6A^2-6B^2-16AB$$
$$y(A,B)=y=-10A^2+10B^2$$
$$z(A,B)=z=4A^2+4B^2$$

Properties

$1.6y(n^2,n+1)-10x(n^2,n+1)=320p_n^5$

$2.6y(n,n+1)-10x(n,n+1)=320t_{3,n}$

$3.6y(n,n^2)-10x(n,n^2)=160cp_{6.n}$

$4.6y(n(n+1),n+2)-10x(n(n+1),n+2)=960p_n^3$

Solutions of Set 2:

$$x(a,b)=x=-5ab$$
$$y(a,b)=y=-2a^2+2b^2-3ab$$
$$z(a,b)=z=a^2+b^2$$

Properties

1) $15[3x(a,b)-5y(a,b)+10t_{4,a}]$ is a nasty number
2) $2z(n,n+1)-x(n,n+1)-6t_{3,n}$ is a perfect square
3) $6[2z(n,n+1)-x(n,n+1)-6t_{3,n}]$ is a nasty number
4) $[y(a,a)+x(a,a)]$ is a perfect square
5) $3[y(a,a)+x(a,a)]$ is a nasty number

Generation of integer solutions

Let (x_0,y_0,z_0) be any given solution of (1)

Formula

Let $x_1=h+8x_0, y_1=h+8y_0, z_1=h-8z_0$

Be the second solution of(1),substituting (13) in (1) and simplifying , we get

$$h=2x_0+2y_0+2z_0$$

The second solution of (1) expressed in the matrix form is

$$(x_1,y_1,z_1)^T=\begin{bmatrix} a_{11}^{(1)}, & a_{12}^{(1)}, & a_{13}^{(1)} \\ a_{21}^{(2)}, & a_{22}^{(2)}, & a_{23}^{(2)} \\ a_{31}^{(3)}, & a_{32}^{(3)}, & a_{33}^{(3)} \end{bmatrix}$$

Where

$$a_{11}^{(1)}=a_{22}^{(1)}=10, a_{12}^{(1)}=a_{21}^{(1)}=a_{31}^{(1)}=a_{32}^{(1)}$$
$$=2, a_{13}^{(1)}=a_{23}^{(1)}=20, a_{11}^{(1)}+a_{12}^{(1)}=a_{33}^{(1)}$$

and T is transpose

Repeating the above process, the general formula for obtaining other integer solutions of (1) is given by

$$(x_n,y_n,z_n)^T=\begin{bmatrix} a_{11}^{(n)}, & a_{12}^{(n)}, & a_{13}^{(n)} \\ a_{21}^{(n)}, & a_{22}^{(n)}, & a_{23}^{(n)} \\ a_{31}^{(n)}, & a_{32}^{(n)}, & a_{33}^{(n)} \end{bmatrix}$$

Where

$$a_{11}^{(n)}+a_{12}^{(n)}=a_{21}^{(n)}+a_{22}^{(n)}=a_{33}^{(n)}, \frac{a_{13}^{(n)}}{a_{11}^{(n)}}=a_{31}^{(n)}$$
$$=a_{32}^{(n)}, a_{31}^{(n)}\times a_{11}^{(n)}=a_{13}^{(n)}=a_{23}^{(n)}$$

Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $5(x^2+y^2)-6xy=20z^2$. As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

References

1. Dickson LE. History of Theory of Numbers and Diophantine Analysis, Vol 2, Dove publications, New York 2005.
2. Mordell LJ. Diophantine Equations" Academic Press, Newyork 1970.
3. Carmicheal RD.The Theory of Numbers and Diophantine Analysis, Dover publications, Newyork, 1959.
4. Gopalan MA,Geetha D. Lattice points on the Hyperboloid of two sheets $x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4$ Impact J Sci Tech 2010; 4:23-32.
5. Gopalan MA,Vidhyalakshmi Kavitha A,Integralpoints on the Homogeneous cone $z^2 = 2x^2 - 7y^2$,The Diophantus J Math 2012;1(2):127-136.
6. Gopalan MA,Vidhyalakshmi S, Sumathi G. Lattice points on the Hyperboloid of one sheet $4z^2 = 2x^2 + 3y^2 - 4$. Diophantus J Math 2012; 1(2):109-115.
7. Gopalan MA, Vidhyalakshmi S, Lakshmi K. Integral points on the Hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$. Diophantus. J Math 2012; 1(2): 99-107.
8. Gopalan MA, Vidhyalakshmi S, Mallika S. Observations on Hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 2$. Bessel. J Math 2012; 2(3): 221-226.
9. Gopalan MA, Vidhyalakshmi S, Usha Rani TR, Mallika S,Integral points on the Homogeneous cone $6z^2 + 3y^2 - 2x^2 = 0$, The Impact. J Sci Tech 2012; 6(1):7-13.
10. Gopalan MA ,Vidhyalakshmi S,Usha Rani TR,Integral points on the non-homogeneous cone $2z^2 + 4xy + 8x - 4z = 0$, Global. Journal of Mathamatics and Mathamatical sciences 2012; 2(1):61-67.
11. Gopalan MA, Vidhyalakshmi S, Uma Rani J, Integral points on the Homogeneous cone $4y^2 + x^2 = 37z^2$, Cayley. J of Math 2013; 2(2):101- 107.
12. Gopalan MA, Vidhyalakshmi S, Kavitha A. Observations on the Hyperboloid of two sheet $7x^2 - 3y^2 = z^2 + z(y - x) + 4$. International. Journal of Latest Research in Science and technology 2013; 2(2): 84-86.
13. Gopalan MA, Sivagami B. Integral points on the homogeneous cone $z^2 = 3x^2 + 6y^2$. IOSR Journal of Mathematics 2013; 8(4):24-29.
14. Gopalan MA, Geetha V. Lattice points on the homogeneous cone, $z^2 = 2x^2 + 8y^2 - 6xy$. Indian. journal of Science, 2013. 2:93-96.
15. Gopalan MA, Vidhyalakshmi S, Umarani J. On the Ternary Quadratic Diophantine Equation $6(x^2 + y^2) - 8xy = 21z^2$. Sch J Eng Tech 2014; 2(2A):108-112.
16. Gopalan MA, Vidhyalakshmi, Santhi J. Lattice points on the homogeneous cone $8(x^2 + y^2) - 15xy = 56z^2$. Sch. Journal of Phy Math Stat 2014; 1(1): 29-32.