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On the ternary quadratic diophantine equation $4(x^2 + y^2) - 7xy = 76z^2$

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Abstract
The ternary quadratic equation given by $4(x^2 + y^2) - 7xy = 76z^2$ is considered and searched for its many different integer solutions. Four different choices of integer solutions to the above equation are presented. A few interesting relations between the solutions and special polygonal numbers are presented.

Keywords: ternary quadratic, integer solutions

1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-16] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $4(x^2 + y^2) - 7xy = 76z^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

Notations

- Polygonal number of rank n with size m

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

- Pronic number of rank n

$$\Pr_n = n(n+1)$$

- Star number of rank n

$$S_n = 6n(n-1) + 1$$

Method of analysis

The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solution is

$$4(x^2 + y^2) - 7xy = 76z^2 \quad (1)$$

Introduction of the linear transformations $(u \neq v \neq 0)$

$$x = u + v, \; y = u - v \quad (2)$$

in (1), leads to

$$u^2 + 15v^2 = 76z^2 \quad (3)$$

Different patterns of solutions of (1) are presented below:

Pattern 1

Write 76 as

$$76 = \left(4 + i2\sqrt{15}\right)\left(4 - i2\sqrt{15}\right) \tag{4}$$

Assume

$$z = z(a,b) = a^2 + 15b^2 \tag{5}$$

Where a and b are non-zero distinct integers.

Using (4), (5) in (3) and employing the method of factorization and equating positive factors, we get

$$u + i\sqrt{15}\,v = \left(4 + i2\sqrt{15}\right)\left(a + i\sqrt{15}\,b\right)^2$$

Equating real and imaginary parts, we get

$$u = 4a^2 - 60b^2 - 60ab$$
$$v = 2a^2 - 30b^2 + 8ab$$

Substituting the values of u and v in (2), we get

$$\left.\begin{array}{l} x = x(a,b) = 6a^2 - 90b^2 - 52ab \\ y = y(a,b) = 2a^2 - 30b^2 - 68ab \end{array}\right\} \tag{6}$$

Thus (5) and (6) represent non zero distinct integral solutions of (1) in two parameters.

Properties

- $x(a,1) - y(a,1) - t_{12,a} + \Pr_a + 60 \equiv 0 \pmod{3}$
- $x(a,1) - y(a,1) - z(a,1) + t_{8,a} - S_a + 76 \equiv 0 \pmod{2}$
- $z(a,a+1) - t_{34,a} - 15 \equiv 0 \pmod{3}$
- $y(a,1) - z(a,1) - t_{4,a} + 45 \equiv 2 \pmod{3}$

Note: 1

Instead of (4), one may write 76 as

$$76 = \frac{\left(17 + i\sqrt{15}\right)\left(17 - i\sqrt{15}\right)}{4} \tag{7}$$

Following the procedure presented in pattern: 1, the corresponding values of x, y and z satisfying (1) are given by

$$x = x(a,b) = 9a^2 - 135b^2 + 2ab$$
$$y = y(a,b) = 8a^2 - 120b^2 - 32ab$$
$$z = z(a,b) = a^2 + 15b^2$$

Pattern 2

Write (3) as

$$u^2 + 15v^2 = 76z^2 * 1 \tag{8}$$

Also, write 1 as

$$1 = \frac{(7 + i\sqrt{15})(7 - i\sqrt{15})}{64} \tag{9}$$

Using (5), (7) and (9) in (8) and employing the method of factorization and equating positive factors, we get

$$u + i\sqrt{15}\,v = \frac{(17 + i\sqrt{15})}{2}\frac{(7 + i\sqrt{15})}{8}(a + i\sqrt{15}\,b)^2$$

Equating real and imaginary parts, we get

$$u = \frac{1}{4}[26a^2 - 390b^2 - 180ab]$$

$$v = \frac{1}{4}[6a^2 - 90b^2 + 52ab]$$

Substituting the values of u and v in (2), we get

$$\left.\begin{array}{l} x = x(a,b) = 8a^2 - 120b^2 - 32ab \\ y = y(a,b) = 5a^2 - 75b^2 - 58ab \end{array}\right\} \tag{10}$$

Thus (10) and (5) represents non zero distinct integral solutions of (1) in two parameters.

Properties

- $x(a, a+1) - y(a, a+1) + t_{86,a} - 26\,\mathrm{Pr}_a + 45 \equiv 1 \pmod 2$
- $x(a,1) + z(a,1) - t_{20,a} + 105 \equiv 0 \pmod 3$
- $y(a,1) + z(a,1) - S_a + 61 \equiv 0 \pmod 2$
- $x(a+1,1) + y(a+1,1) + z(a, a+1) - t_{30,a} + 90\,\mathrm{Pr}_a + 166 \equiv 1 \pmod 5$

Note 2

Instead of (9), one may write 1 as

$$1 = \frac{(1 + i\sqrt{15})(1 - i\sqrt{15})}{16} \tag{11}$$

and instead of (7), choose (4) and repeating the process as in Pattern: 2, the corresponding non-zero distinct integer solutions of (1) are given by

$$x = x(a,b) = -5a^2 + 75b^2 - 58ab$$

$$y = y(a,b) = -8a^2 + 120b^2 - 32ab$$

$$z = z(a,b) = a^2 + 15b^2$$

Pattern 3

It is worth to note that (3) can be written as

$$76z^2 - u^2 = 15v^2 \tag{12}$$

Assume $v = 76a^2 - b^2$ (13)

Write 15 as

$$15 = \frac{\left(\sqrt{76}+4\right)\left(\sqrt{76}-4\right)}{4} \qquad (14)$$

Using (13), (14) in (12) and employing the method of factorization and equating positive factors, we get

$$\left(\sqrt{76}\,z+u\right)=\left(\sqrt{76}\,a+b\right)^2\frac{\left(\sqrt{76}+4\right)}{2}$$

Equating rational and irrational parts, we get

$$u = 152a^2+2b^2+76ab$$
$$z = \frac{1}{2}\left[76a^2+b^2+8ab\right]$$

Substituting the values u and v in (2), we get

$$x = 228a^2+b^2+76ab$$
$$y = 76a^2+3b^2+76ab$$

Replacing b by $2b$, we get

$$x = x(a,b) = 228a^2+4b^2+152ab$$
$$y = y(a,b) = 76a^2+12b^2+152ab$$
$$z = z(a,b) = 38a^2+2b^2+8ab$$

which represents non zero distinct integral solutions of (1) in two parameters.

Properties

- $x(a,1)-y(a,1)-152t_{4,a}+8=0$
- $x(1,b)-y(1,b)+z(1,b)+t_{14,6}-190\equiv 0\pmod{3}$
- $y(a,a+1)-z(a,a+1)-144\Pr_a-t_{44,a}-10\equiv 0\pmod{2}$

Pattern 4

Assume

$$u = 2U,\ v = 2V \qquad (15)$$

Using (15) in (3), we get

$$(U+2z)(U-2z)=15(z+V)(z-V) \qquad (16)$$

Factorizing (16), we have

$$\frac{U+2z}{z-V}=\frac{15(z+V)}{U-2z}=\frac{m}{n},\ n\neq 0 \qquad (17)$$

which is equivalent to the system of double equations

$$mV + nU + z(2n - m) = 0$$

$$15nV - mU + z(15n + 2m) = 0$$

Applying the method of cross multiplication, we have

$$U = 2m^2 - 30n^2 + 30mn \ , V = m^2 - 15n^2 - 4mn \ , z = m^2 + 15n^2$$

In view of (15), we get

$$u = 4m^2 - 60n^2 + 60mn \ , v = 2m^2 - 30n^2 - 8mn$$

Substituting the values of u and v in (2), we get

$$\left.\begin{aligned} x &= x(m,n) = 6m^2 - 90n^2 + 52mn \\ y &= y(m,n) = 2m^2 - 30n^2 + 68mn \\ z &= z(m,n) = m^2 + 15n^2 \end{aligned}\right\} \quad (18)$$

Thus (18) represents the integer solutions to (1).

Properties

- $x(m,1) - y(m,1) + z(m,1) - t_{12,m} + 45 \equiv 0 \pmod{3}$
- $x(m,1) - S_m + 91 \equiv 0 \pmod{2}$
- $y(m, m+1) + z(m, m+1) - 68\,\mathrm{Pr}_m + t_{26,m} + 15 \equiv 1 \pmod{2}$
- $y(m, m+1) - 68\,\mathrm{Pr}_m + t_{58,m} + 30 \equiv 0 \pmod{3}$

Remark
It is worth to note that (17) is also represented in the following ways:

1. $\dfrac{U + 2z}{15(z + V)} = \dfrac{z - V}{U - 2z} = \dfrac{m}{n}, n \neq 0$
2. $\dfrac{U + 2z}{5(z + V)} = \dfrac{3(z - V)}{U - 2z} = \dfrac{m}{n}, n \neq 0$
3. $\dfrac{U + 2z}{3(z - V)} = \dfrac{5(z + V)}{U - 2z} = \dfrac{m}{n}, n \neq 0$

By introducing the above representations instead of (17), one may obtain different sets of distinct integer solutions to (1) and their corresponding properties.

Solutions for Set: (i)

$$x = x(m,n) = 30m^2 - 2n^2 + 68mn$$

$$y = y(m,n) = 90m^2 - 6n^2 + 52mn$$

$$z = z(m,n) = 15m^2 + n^2$$

Properties

$$y(1,n) - x(1,n) - z(1,n) + t_{12,n} - 45 \equiv 0 \pmod{5}$$

$$x(m, m+1) + 68\ \Pr_m - S_m - t_{46,m} \equiv 1 \pmod{3}$$

Solutions For Set: (ii)

$$x = x(m,n) = 10m^2 - 6n^2 + 68mn$$
$$y = y(m,n) = 30m^2 - 18n^2 + 52mn$$
$$z = z(m,n) = 5m^2 + 3n^2$$

Properties

- $y(m,1) - 2x(m,1) - t_{10,m} - S_m + 7 \equiv 0 \pmod{5}$
- $y(m,1) - x(m,1) - z(m,1) - t_{32,m} + 15 \equiv 0 \pmod{2}$
- $x(m, m+1) - 68\ \Pr_m - t_{10,m} + 6 \equiv 0 \pmod{3}$

Solutions For Set: (iii)

$$x = x(m,n) = 10m^2 - 6n^2 + 68mn$$
$$y = y(m,n) = 30m^2 - 18n^2 + 52mn$$
$$z = z(m,n) = 5m^2 + 3n^2$$

Properties

- $y(m,1) - 2x(m,1) - t_{10,m} - S_m + 7 \equiv 0 \pmod{5}$
- $y(m,1) - x(m,1) - z(m,1) - t_{32,m} + 15 \equiv 0 \pmod{2}$
- $x(m, m+1) - 68\ \Pr_m - t_{10,m} + 6 \equiv 0 \pmod{3}$

Conclusion

In this paper, we have presented infinitely many non-zero distinct integer solutions to the ternary quadratic equation $4(x^2 + y^2) - 7xy = 76z^2$ Representing a homogeneous cone. As diophantine equations are rich in variety.

To conclude, one may search for other forms of three dimensional surfaces, namely, non-homogeneous cone, paraboloid, ellipsoid, hyperboloid, hyperbolic paraboloid and so on for finding integral points on them and corresponding properties

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